

# Lebesgue constants for polyhedral sets and polynomial interpolation on Lissajous-Chebyshev nodes

Peter Dencker<sup>1)</sup>, Wolfgang Erb<sup>1),a)</sup>,  
 Yurii Kolomoitsev<sup>1),2),b),\*</sup>, Tetiana Lomako<sup>1),2),b)</sup>

dencker@math.uni-luebeck.de, erb@math.uni-luebeck.de,  
 kolomus1@mail.ru, tlomako@yandex.ru

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## Abstract

To analyze the absolute condition number of multivariate polynomial interpolation on Lissajous-Chebyshev node points, we derive upper and lower bounds for the respective Lebesgue constant. The proof is based on a relation between the Lebesgue constant for the polynomial interpolation problem and the Lebesgue constant linked to the polyhedral partial sums of Fourier series. The magnitude of the obtained bounds is determined by a product of logarithms of the side lengths of the considered polyhedral sets and shows the same behavior as the magnitude of the Lebesgue constant for polynomial interpolation on the tensor product Chebyshev grid.

**Keywords:** interpolation, Lissajous-Chebyshev nodes, Lebesgue constants, polyhedra

<sup>1)</sup>Institut für Mathematik, Universität zu Lübeck.

<sup>2)</sup>Institute of Applied Mathematics and Mechanics, Academy of Sciences of Ukraine.

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\*Corresponding author.

# 1 Introduction

In [9, 13, 14], the multivariate polynomial interpolation on the Lissajous-Chebyshev node points  $\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$  (see (3.1)) is studied. Here, the parameters  $\underline{\kappa} \in \mathbb{Z}^d$  and  $\epsilon \in \{1, 2\}$  determine the underlying types of Lissajous curves, and the vector

$$\underline{\mathbf{n}} = (n_1, \dots, n_d) \in \mathbb{N}^d \text{ with pairwise relatively prime entries } n_1, \dots, n_d \in \mathbb{N} \quad (1.1)$$

describes the frequencies of the Lissajous curve with respect to the coordinate axis. These considerations are motivated by applications in a novel medical imaging modality called Magnetic Particle Imaging (see [14, 15]). The problem reads as follows:

For the node points  $\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$  and a function  $f : [-1, 1]^d \rightarrow \mathbb{R}$  with values  $f(\underline{\mathbf{z}})$  at the node points  $\underline{\mathbf{z}} = (z_1, \dots, z_d) \in \underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$ , we aim to find a  $d$ -variate interpolation polynomial  $P_{\underline{\kappa}}^{(\epsilon \mathbf{n})} f$  such that

$$P_{\underline{\kappa}}^{(\epsilon \mathbf{n})} f(\underline{\mathbf{z}}) = f(\underline{\mathbf{z}}) \quad \text{for all } \underline{\mathbf{z}} \in \underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}. \quad (1.2)$$

It was shown in [9] that the interpolation problem (1.2) has a unique solution in the polynomial space  $\Pi_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$  that is linearly spanned by all  $d$ -variate Chebyshev polynomials  $T_{\underline{\gamma}}$ , where  $\underline{\gamma}$  is an element of the index set

$$\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \mathbf{n})} = \left\{ \underline{\gamma} \in \mathbb{N}_0^d \left| \begin{array}{l} \gamma_i/n_i < \epsilon \quad \forall i \in \{1, \dots, d\}, \\ \gamma_i/n_i + \gamma_j/n_j \leq \epsilon \quad \forall i, j \text{ with } i \neq j, \\ \gamma_i/n_i + \gamma_j/n_j < \epsilon \quad \forall i, j \text{ with } \kappa_i \not\equiv \kappa_j \pmod{2} \end{array} \right. \right\} \cup \{(0, \dots, 0, \epsilon n_d)\}.$$

The nodes  $\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$ , the Chebyshev polynomials  $T_{\underline{\gamma}}$ , and the interpolation problem will be recapitulated in more detail in Section 3 of this article.

The absolute condition number of the interpolation problem (1.2) with respect to the uniform norm  $\|f\|_{\infty} = \operatorname{ess\,sup}_{\underline{\mathbf{x}} \in [-1, 1]^d} |f(\underline{\mathbf{x}})|$  (see [10, p. 26]) is given by the *Lebesgue constant of the interpolation problem*, i.e.

$$\Lambda_{\underline{\kappa}}^{(\epsilon \mathbf{n})} = \sup_{f \in C([-1, 1]^d): \|f\|_{\infty} \leq 1} \|P_{\underline{\kappa}}^{(\epsilon \mathbf{n})} f\|_{\infty}. \quad (1.3)$$

Beside its relation to the numerical stability of the interpolation problem (1.2), the Lebesgue constant (1.3) is also an essential tool for the investigation of the approximation error  $\|f - P_{\underline{\kappa}}^{(\epsilon \mathbf{n})} f\|_{\infty}$ .

A main goal of this article is to provide for all  $\underline{\mathbf{n}}$  satisfying (1.1) asymptotic upper and lower bounds for the Lebesgue constants (1.3) in the sense of (1.13). The corresponding result in Theorem 3.4 states

$$\Lambda_{\underline{\kappa}}^{(\epsilon \mathbf{n})} \asymp \prod_{i=1}^d \ln(n_i + 1). \quad (1.4)$$

In particular, the upper and lower estimates have asymptotically the same magnitude as the Lebesgue constants for polynomial interpolation on the tensor product Chebyshev grid (see [6]). Thus, the interpolation problem (1.2) in  $\Pi_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$  is asymptotically as well-conditioned as the mentioned tensor product case. The upper estimate in (1.4) of the Lebesgue constant  $\Lambda_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$  is further used in Corollary 3.5 to formulate a multivariate error estimate and an example of a Dini-Lipschitz-type condition for the uniform convergence of the interpolation polynomials  $P_{\underline{\kappa}}^{(\epsilon \mathbf{n})} f$ .

In the bivariate setting, the obtained results are generalizations of the corresponding results for the Padua points in [5, 7, 8] and improvements of estimates given in [12].

We sketch our program for the proof of (1.4). For a finite set  $\underline{\Gamma} \subset \mathbb{Z}^d$ , the *Lebesgue constant*  $L(\underline{\Gamma})$  related to partial Fourier series is defined as

$$L(\underline{\Gamma}) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \sum_{\underline{\gamma} \in \underline{\Gamma}} e^{i(\underline{\gamma}, \underline{t})} \right| d\underline{t},$$

where

$$(\underline{\gamma}, \underline{t}) = \sum_{i=1}^d \gamma_i t_i.$$

To obtain the upper and lower bounds for (1.3), our strategy in the proof of Theorem 3.4 below consists in establishing the relations

$$\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})} \lesssim L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}) + \prod_{i=1}^d \ln(n_i + 1), \quad L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}) \lesssim \Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})} \quad (1.5)$$

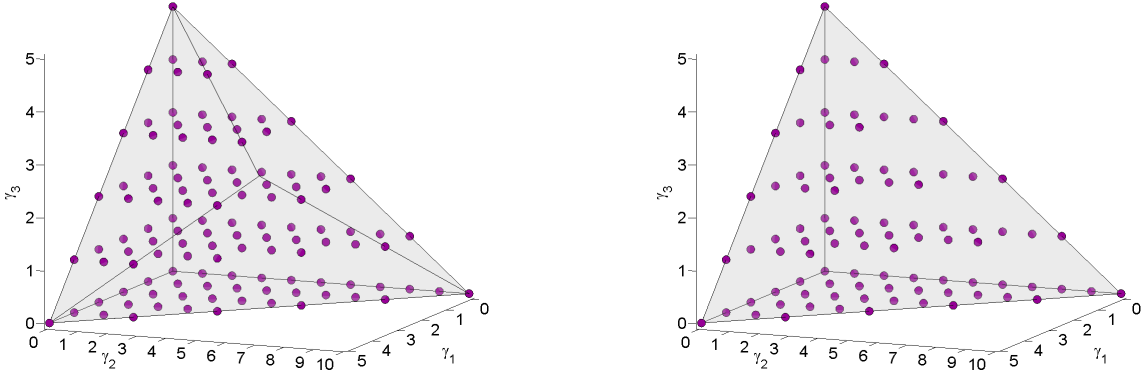
between  $\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})}$  and the Lebesgue constants  $L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*})$  of the symmetrized sets  $\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}$ . Here and in the following, for every  $\underline{\Gamma} \subset \mathbb{Z}^d$  its symmetrization  $\underline{\Gamma}^*$  is defined as

$$\underline{\Gamma}^* = \left\{ \underline{\gamma} \in \mathbb{Z}^d \mid (|\gamma_1|, \dots, |\gamma_d|) \in \underline{\Gamma} \right\}. \quad (1.6)$$

Using the methods developed in Section 2, Corollary 3.3 states that

$$L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}) \asymp \prod_{i=1}^d \ln(n_i + 1). \quad (1.7)$$

Then, combining (1.5) and (1.7) yields (1.4).



**Figure 1** Illustration of the sets  $\bar{\Gamma}^{(m)}$  (left) and  $\underline{\Sigma}_1^{(m)}$  (right) for  $\underline{m} = (5, 10, 5)$ .

The technically more sophisticated part of the sketched program is the proof of (1.7). The used methods are developed in Section 2. Therein we consider the sets

$$\bar{\Gamma}^{(m)} = \left\{ \underline{\gamma} \in \mathbb{N}_0^d \mid \begin{array}{l} \gamma_i/m_i \leq 1 \quad \forall i \in \{1, \dots, d\}, \\ \gamma_i/m_i + \gamma_j/m_j \leq 1 \quad \forall i, j \text{ with } i \neq j \end{array} \right\}$$

and its symmetrizations  $\bar{\Gamma}^{(m),*}$  according to (1.6). The used methods for these sets are templates for the corresponding methods for the sets  $\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}$  and  $\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}$ , respectively. It turns out that similar methods can be used to estimate for fixed rational  $r > 0$  the Lebesgue constants of families of sets  $\underline{\Sigma}_r^{(m)}$  and its symmetrizations  $\underline{\Sigma}_r^{(m),*}$ , where

$$\underline{\Sigma}_r^{(m)} = \left\{ \underline{\gamma} \in \mathbb{N}_0^d \mid \sum_{i=1}^d \frac{\gamma_i}{m_i} \leq r \right\}.$$

Note that  $\overline{\mathbf{I}}^{(m_1, m_2),*} = \underline{\Sigma}_1^{(m_1, m_2),*}$  in dimension  $d = 2$ . Sets of this kind are illustrated in Figure 1 and are of interest since they might be used as elementary building blocks for more complex polyhedra. Further, our results could be useful for the investigation of generalizations of the triangular partial Fourier series in [25].

Estimates of the Lebesgue constant  $L(\mathbf{I})$  for various types of sets  $\mathbf{I}$  are extensively investigated in the literature. An overview about the state of the art can be found in the survey article [17]. Since we are dealing with sets having a polyhedral structure, estimates of the Lebesgue constants for those sets are particularly interesting for us. If  $\underline{\mathbf{E}}$  is a fixed  $d$ -dimensional convex polyhedron containing the origin, then it is well-known (see [3, 4, 20, 24, 27, 28]) that for all real  $m \geq 1$  we have

$$L(m\underline{\mathbf{E}} \cap \mathbb{Z}^d) \asymp (\ln(m+1))^d.$$

In this work, we want to refine this asymptotic result for special  $d$ -dimensional polyhedra in which integer-valued directional dilation parameters  $m_1, \dots, m_d \in \mathbb{N}$  are given. An example for different directional parameters is the case of rectangular sets  $\underline{\mathbf{R}}^{(\underline{m})} = [0, m_1] \times \dots \times [0, m_d]$ . In this case, for all  $m_1, \dots, m_d \geq 1$ , we have

$$L(\underline{\mathbf{R}}^{(\underline{m})} \cap \mathbb{Z}^d) \asymp L(\underline{\mathbf{R}}^{(\underline{m}),*} \cap \mathbb{Z}^d) \asymp \prod_{i=1}^d \ln(m_i + 1). \quad (1.8)$$

This immediately follows from the well-known one-dimensional case (see [1]).

The starting points for our investigations of  $L(\mathbf{I})$  are two estimates of the Lebesgue constant given in [27] and [28]. In [27, Theorem 2] it is stated that for all polyhedra  $\underline{\mathbf{E}} \in \mathbb{R}^2$  with  $n$  edges, we have the uniform upper bound

$$L(\underline{\mathbf{E}} \cap \mathbb{Z}^2) \lesssim n (\ln \text{diam}(\underline{\mathbf{E}}))^2. \quad (1.9)$$

Further, it is shown in [28] that for all convex sets  $\underline{\mathbf{E}} \in \mathbb{R}^d$  containing a ball with radius  $r \geq 1$  we have the lower bound

$$L(\underline{\mathbf{E}} \cap \mathbb{Z}^d) \gtrsim (\ln(r+1))^d. \quad (1.10)$$

Combining (1.9) and (1.10) yields that for all real  $m_1, m_2 \geq 1$  we have the uniform upper and lower bound

$$(\ln(\min(m_1, m_2) + 1))^2 \lesssim L(\overline{\mathbf{I}}^{(m_1, m_2),*}) \lesssim (\ln(\max(m_1, m_2) + 1))^2. \quad (1.11)$$

A special case of our result (see Theorem 2.1) is that for all positive integers  $m_1, m_2$  we can improve (1.11) to  $L(\overline{\mathbf{I}}^{(m_1, m_2),*}) \asymp \ln(m_1 + 1) \ln(m_2 + 1)$ . Under the strongly restrictive condition that  $m_2$  is a multiple of  $m_1$  this result appears already in [16].

In general, Theorem 2.1 states that for all  $\underline{m} \in \mathbb{N}^d$  we have

$$L(\overline{\mathbf{I}}^{(\underline{m})}) \asymp L(\overline{\mathbf{I}}^{(\underline{m}),*}) \asymp \prod_{i=1}^d \ln(m_i + 1).$$

Thus, the magnitude of the uniform upper and lower bounds is the same as in the rectangular case (1.8). Similarly, Theorem 2.3 states that for a fixed  $r \in \mathbb{Q}$ ,  $r > 0$ , and all  $\underline{m} \in \mathbb{N}^d$  we have

$$L(\underline{\Sigma}_r^{(\underline{m})}) \asymp L(\underline{\Sigma}_r^{(\underline{m}),*}) \asymp \prod_{i=1}^d \ln(m_i + 1).$$

In Section 2, we consider also another type of polyhedral sets given by

$$\Xi_{(r,s)}^{(\underline{m})} = \left\{ \gamma \in \mathbb{Z}^d \mid r \leq \frac{\gamma_d}{m_d} \leq \dots \leq \frac{\gamma_2}{m_2} \leq \frac{\gamma_1}{m_1} \leq s \right\}. \quad (1.12)$$

For fixed  $r, s \in \mathbb{R}$  and all positive integers  $m_1, \dots, m_d \in \mathbb{N}$  an uniform upper bound  $L(\Xi_{(r,s)}^{(\underline{m})}) \lesssim \prod_{i=1}^d \ln(m_i + 1)$  is established for the corresponding Lebesgue constant in Theorem 2.2. The proof of the upper bound of the Lebesgue constants for the polyhedral sets  $\underline{\Gamma}^{(\underline{m})}$  and  $\underline{\Gamma}^{(\underline{m}),*}$  uses slightly generalized versions (see (2.40)) of the polyhedral sets (1.12) as building blocks. The techniques presented in the proofs of Section 2 are interesting in their own regard and might be as well useful for the consideration of other types of polyhedral sets.

## General notation

For  $x \in \mathbb{R}$ , we use  $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$ ,  $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$  and denote

$$\llbracket x \rrbracket = x - \lfloor x \rfloor, \quad \lceil x \rceil = \lceil x \rceil - x.$$

Let  $f$  and  $g$  be real functions on a set  $X$ . The notation

$$f(x) \lesssim g(x) \text{ for all } x \in X$$

has by definition the following meaning:

There exists a constant  $C > 0$  such that  $f(x) \leq Cg(x)$  for all  $x \in X$ .

Furthermore, we write

$$f(x) \asymp g(x) \text{ for all } x \in X, \quad (1.13)$$

if for all  $x \in X$  we have both,  $f(x) \lesssim g(x)$  and  $g(x) \lesssim f(x)$ .

We write  $\underline{x} = (x_1, \dots, x_d)$  for elements of the euclidean space  $\mathbb{R}^d$  with fixed  $d \in \mathbb{N}$ . For  $a, b \in \mathbb{R}$ ,  $a < b$ , and  $1 \leq p < \infty$  and Lebesgue-measurable  $f : [a, b]^d \rightarrow \mathbb{R}$ , we set

$$\|f\|_{L^p([a,b]^d)} = \left( \frac{1}{(b-a)^d} \int_{[a,b]^d} |f(\underline{t})|^p d\underline{t} \right)^{1/p},$$

and for Lebesgue-measurable functions  $f : [-1, 1]^d \rightarrow \mathbb{R}$ , and  $1 \leq p < \infty$ , we define

$$\|f\|_{w_d, p} = \left( \frac{1}{\pi^d} \int_{[-1, 1]^d} |f(\underline{x})|^p w_d(\underline{x}) d\underline{x} \right)^{1/p}, \quad w_d(\underline{x}) = \prod_{i=1}^d \frac{1}{\sqrt{1 - x_i^2}}.$$

## 2 Lebesgue constants for polyhedral partial sums of Fourier series

We summarize the main results of this section.

**Theorem 2.1** *For all  $\underline{m} \in \mathbb{N}^d$ , we have*

$$L(\underline{\Gamma}^{(\underline{m})}) \asymp L(\underline{\Gamma}^{(\underline{m}),*}) \asymp \prod_{i=1}^d \ln(m_i + 1).$$

In Section 3, we will apply this theorem to obtain estimates of the Lebesgue constant for the interpolation problem on the Lissajous-Chebyshev nodes. To prove Theorem 2.1 we will use the following statement which is also interesting by itself.

**Theorem 2.2** *Let  $r, s \in \mathbb{R}$ ,  $0 \leq r < s$ , be fixed. For all  $\underline{m} \in \mathbb{N}^d$ , we have*

$$L\left(\underline{\Xi}_{(r,s)}^{(\underline{m})}\right) \lesssim \prod_{i=1}^d \ln(m_i + 1). \quad (2.1)$$

Further, let us consider the sets  $\underline{\Sigma}_r^{(\underline{m})}$  and  $\underline{\Sigma}_r^{(\underline{m})*}$ . These sets can be considered as another possible generalization of the sets considered in [27] for  $\underline{m} \in \mathbb{Z}^d$ , and they are interesting since they may be used as building blocks for certain polyhedra.

**Theorem 2.3** *Let  $r \in \mathbb{Q}$ ,  $r > 0$ , be fixed. For all  $\underline{m} \in \mathbb{N}^d$ , we have*

$$L\left(\underline{\Sigma}_r^{(\underline{m})}\right) \asymp L\left(\underline{\Sigma}_r^{(\underline{m})*}\right) \asymp \prod_{i=1}^d \ln(m_i + 1). \quad (2.2)$$

The proofs of these results are given in Subsections 2.2, 2.1, and 2.3, respectively.

## 2.1 Proof of Theorem 2.2

Let us first formulate and prove several auxiliary statements.

For  $d \in \mathbb{N}$ ,  $\underline{m} \in (0, \infty)^d$  and  $r, s \in \mathbb{R}$ ,  $r < s$ , we set

$$D_{(r,s)}^{(\underline{m})}(\underline{t}) = \sum_{\underline{\gamma} \in \underline{\Xi}_{(r,s)}^{(\underline{m})}} e^{i(\underline{\gamma}, \underline{t})}.$$

Let  $d \geq 2$  everywhere below. For  $1 \leq k \leq d$ , we denote

$$D_{k,(r,s)}^{\circ,(\underline{m})}(\underline{t}) = D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}(t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_d).$$

For  $2 \leq k \leq d$ , we introduce

$$D_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}(t_1, \dots, t_{k-2}, t_{k-1} + t_k m_k / m_{k-1}, t_{k+1}, \dots, t_d),$$

$$\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = D_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) - D_{k,(r,s)}^{\circ,(\underline{m})}(\underline{t}), \quad (2.3)$$

and

$$F_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = \frac{e^{it_k}}{e^{it_k} - 1} \sum^{\circ k} e^{i(\gamma_1 t_1 + \dots + \gamma_{k-2} t_{k-2} + \gamma_{k+1} t_{k+1} + \dots + \gamma_d t_d)} \\ \times e^{i\gamma_{k-1}(t_{k-1} + t_k m_k / m_{k-1})} (e^{-i\lfloor \gamma_{k-1} m_k / m_{k-1} \rfloor t_k} - 1).$$

Here and in the following,

$$\sum^{\circ k} \text{ means the sum over } (\gamma_1, \dots, \gamma_{k-1}, \gamma_{k+1}, \dots, \gamma_d) \in \underline{\Xi}_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}. \quad (2.4)$$

In the special case  $k = d$ , for simplicity, we denote

$$D_{(r,s)}^{\circ,(\underline{m})}(\underline{t}) = D_{d,(r,s)}^{\circ,(\underline{m})}(\underline{t}), \quad D_{(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = D_{d,(r,s)}^{\sharp,(\underline{m})}(\underline{t}),$$

$$\Delta_{(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = \Delta_{d,(r,s)}^{\sharp,(\underline{m})}(\underline{t}), \quad F_{(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = F_{d,(r,s)}^{\sharp,(\underline{m})}(\underline{t}),$$

and

$$G_{(r,s)}^{(\underline{m})}(\underline{t}) = \frac{1}{e^{it_d} - 1} \left( \Delta_{(r,s)}^{\sharp,(\underline{m})}(\underline{t}) - (e^{i\lceil r m_d \rceil t_d} - 1) D_{(r,s)}^{\circ,(\underline{m})}(\underline{t}) \right).$$

**Proposition 2.4** Let  $\underline{m} \in (0, \infty)^d$  and  $r, s \in \mathbb{R}$ ,  $r < s$ . Then

$$D_{(r,s)}^{(\underline{m})}(\underline{t}) = G_{(r,s)}^{(\underline{m})}(\underline{t}) + D_{(r,s)}^{\sharp,(\underline{m})}(\underline{t}) + F_{(r,s)}^{\sharp,(\underline{m})}(\underline{t}). \quad (2.5)$$

*Proof.* First, we show that

$$\begin{aligned} D_{(r,s)}^{(m_{d-1}, m_d)}(t_{d-1}, t_d) &= G_{(r,s)}^{(m_{d-1}, m_d)}(t_{d-1}, t_d) + D_{(r,s)}^{\sharp, (m_{d-1}, m_d)}(t_{d-1}, t_d) \\ &\quad + F_{(r,s)}^{\sharp, (m_{d-1}, m_d)}(t_{d-1}, t_d). \end{aligned} \quad (2.6)$$

Indeed, we have that  $G_{(r,s)}^{(m_{d-1}, m_d)}(t_{d-1}, t_d) + D_{(r,s)}^{\sharp, (m_{d-1}, m_d)}(t_{d-1}, t_d)$  equals

$$\begin{aligned} &\frac{1}{e^{it_d} - 1} \Delta_{(r,s)}^{\sharp, (m_{d-1}, m_d)}(t_{d-1}, t_d) - \frac{e^{i\lceil rm_d \rceil t_d} - 1}{e^{it_d} - 1} D_{(r,s)}^{(m_{d-1})}(t_{d-1}) + D_{(r,s)}^{(m_{d-1})}(t_{d-1} + t_d m_d / m_{d-1}) \\ &= \frac{1}{e^{it_d} - 1} \left( e^{it_d} D_{(r,s)}^{(m_{d-1})}(t_{d-1} + t_d m_d / m_{d-1}) - e^{i\lceil rm_d \rceil t_d} D_{(r,s)}^{(m_{d-1})}(t_{d-1}) \right), \end{aligned}$$

and

$$F_{(r,s)}^{\sharp, (m_{d-1}, m_d)}(t_{d-1}, t_d) = \frac{e^{it_d}}{e^{it_d} - 1} \sum_{\gamma_{d-1}=\lceil rm_{d-1} \rceil}^{\lfloor sm_{d-1} \rfloor} e^{i\gamma_{d-1}(t_{d-1} + t_d m_d / m_{d-1})} \left( e^{-i\lfloor \gamma_{d-1} m_d / m_{d-1} \rfloor t_d} - 1 \right).$$

Now, (2.6) follows from

$$e^{i\gamma_{d-1} t_{d-1}} e^{i(\lfloor \gamma_{d-1} m_d / m_{d-1} \rfloor + 1) t_d} = e^{it_d} e^{i\gamma_{d-1}(t_{d-1} + t_d m_d / m_{d-1})} e^{-i\lfloor \gamma_{d-1} m_d / m_{d-1} \rfloor t_d}.$$

For the functions corresponding to the symbols  $S \in \{D, D^\circ, D^\sharp, \Delta^\sharp, G, F^\sharp\}$ , we have the descending recursive relation

$$S_{(r,s)}^{(m_i, \dots, m_d)}(t_i, \dots, t_d) = \sum_{\gamma_i=\lceil rm_i \rceil}^{\lfloor sm_i \rfloor} e^{i\gamma_i t_i} S_{(r, (\gamma_i/m_i))}^{(m_{i+1}, \dots, m_d)}(t_{i+1}, \dots, t_d), \quad 1 \leq i \leq d-2. \quad (2.7)$$

Equality (2.6) means that we have (2.5) with  $(m_{d-1}, m_d)$  in place of  $\underline{m}$ , and  $(t_{d-1}, t_d)$  in place of  $\underline{t}$ . Thus, induction argument using the relation (2.7) for  $S \in \{D, G, D^\sharp, F^\sharp\}$  yield that for  $i \in \{d-2, \dots, 2, 1\}$  we have (2.5) with  $(m_i, \dots, m_d)$  in place of  $\underline{m}$ , and  $(t_i, \dots, t_d)$  in place of  $\underline{t}$ . In particular, for  $i = 1$ , we have (2.5).  $\square$

Next, for  $1 \leq k \leq d-1$ , we introduce

$$D_{k,(r,s)}^{b,(\underline{m})}(\underline{t}) = D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}(t_1, \dots, t_{k-1}, t_{k+1} + t_k m_k / m_{k+1}, t_{k+2}, \dots, t_d),$$

and

$$\Delta_{k,(r,s)}^{b,(\underline{m})}(\underline{t}) = D_{k,(r,s)}^{b,(\underline{m})}(\underline{t}) - D_{k,(r,s)}^{\circ,(\underline{t})}(\underline{t}),$$

and, using (2.4), we set

$$\begin{aligned} F_{k,(r,s)}^{b,(\underline{m})}(\underline{t}) &= \frac{1}{e^{it_k} - 1} \sum^{\circ k} e^{i(\gamma_1 t_1 + \dots + \gamma_{k-1} t_{k-1} + \gamma_{k+2} t_{k+2} + \dots + \gamma_d t_d)} \\ &\quad \times e^{i\gamma_{k+1}(t_{k+1} + t_k m_k / m_{k+1})} \left( e^{i\lfloor \gamma_{k+1} m_k / m_{k+1} \rfloor t_k} - 1 \right). \end{aligned}$$

We also denote

$$G_{k,(r,s)}^{(\underline{m})}(\underline{t}) = \frac{1}{e^{it_k} - 1} \begin{cases} (e^{i(\lfloor sm_1 \rfloor + 1) t_1} - 1) D_{1,(r,s)}^{\circ,(\underline{m})}(\underline{t}) - \Delta_{1,(r,s)}^{b,(\underline{m})}(\underline{t}) & \text{if } k = 1, \\ \Delta_{d,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) - (e^{i\lceil rm_d \rceil t_d} - 1) D_{d,(r,s)}^{\circ,(\underline{m})}(\underline{t}) & \text{if } k = d, \\ \Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) - \Delta_{k,(r,s)}^{b,(\underline{m})}(\underline{t}) & \text{if } 2 \leq k \leq d-1, \end{cases}$$

and

$$F_{k,(r,s)}^{(\underline{m})}(\underline{t}) = \begin{cases} -F_{1,(r,s)}^{b,(\underline{m})}(\underline{t}) & \text{if } k = 1, \\ F_{d,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) & \text{if } k = d, \\ F_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) - F_{k,(r,s)}^{b,(\underline{m})}(\underline{t}) & \text{if } 2 \leq k \leq d-1, \end{cases}$$

and

$$H_{k,(r,s)}^{(\underline{m})}(\underline{t}) = \begin{cases} 0 & \text{if } k = 1, \\ D_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) & \text{if } 2 \leq k \leq d. \end{cases}$$

**Proposition 2.5** Let  $\underline{m} \in (0, \infty)^d$ ,  $r, s \in \mathbb{R}$ ,  $r < s$ , and  $k \in \{1, \dots, d\}$ . Then

$$D_{k,(r,s)}^{(\underline{m})}(\underline{t}) = G_{k,(r,s)}^{(\underline{m})}(\underline{t}) + H_{k,(r,s)}^{(\underline{m})}(\underline{t}) + F_{k,(r,s)}^{(\underline{m})}(\underline{t}). \quad (2.8)$$

*Proof.* In the case  $k = d$ , the equality (2.8) is proved in Proposition 2.4. Let us consider the case  $2 \leq k \leq d-1$ . By the definitions of  $G_{d,(r,s)}^{(\underline{m})}(\underline{t})$ ,  $\Delta_{d,(r,s)}^{\sharp,(\underline{m})}(\underline{t})$  with  $k$  instead of  $d$ ,  $(m_1, \dots, m_k)$  instead of  $\underline{m}$ , and  $(t_1, \dots, t_k)$  instead of  $\underline{t}$ , we have

$$\begin{aligned} G_{k,(r,s)}^{(m_1, \dots, m_k)}(t_1, \dots, t_k) &= \frac{1}{e^{it_k} - 1} \Delta_{k,(r,s)}^{\sharp, (m_1, \dots, m_k)}(t_1, \dots, t_k) - \frac{e^{i[r m_k] t_k} - 1}{e^{it_k} - 1} D_{k,(r,s)}^{\circ, (\underline{m})}(t_1, \dots, t_k) \\ &= \frac{1}{e^{it_k} - 1} D_{k,(r,s)}^{\sharp, (m_1, \dots, m_k)}(t_1, \dots, t_k) - \frac{e^{i[r m_k] t_k}}{e^{it_k} - 1} D_{k,(r,s)}^{\circ, (\underline{m})}(t_1, \dots, t_k). \end{aligned}$$

At the same time, Proposition 2.4 with  $(m_1, \dots, m_k)$  instead of  $\underline{m}$  and  $(t_1, \dots, t_k)$  instead of  $\underline{t}$  gives the equality

$$\begin{aligned} D_{k,(r,s)}^{(m_1, \dots, m_k)}(t_1, \dots, t_k) &+ \frac{e^{i[r m_k] t_k}}{e^{it_k} - 1} D_{k,(r,s)}^{\circ, (\underline{m})}(t_1, \dots, t_k) \\ &= \frac{e^{it_k}}{e^{it_k} - 1} D_{k,(r,s)}^{\sharp, (m_1, \dots, m_k)}(t_1, \dots, t_k) + F_{k,(r,s)}^{\sharp, (m_1, \dots, m_k)}(t_1, \dots, t_k). \end{aligned} \quad (2.9)$$

For the functions corresponding to the symbols  $S \in \{D, G, H, F\}$ , we have the ascending recursion relation

$$S_{k,(r,s)}^{(m_1, \dots, m_i)}(t_1, \dots, t_i) = \sum_{\gamma_i = [r m_i]}^{[s m_i]} e^{i \gamma_i t_i} S_{k, ((\gamma_i / m_i), s)}^{(m_1, \dots, m_{i-1})}(t_1, \dots, t_{i-1}), \quad k+2 \leq i \leq d. \quad (2.10)$$

Below, we will show that (2.8) is satisfied with  $(m_1, \dots, m_{k+1})$  instead of  $\underline{m}$  and  $(t_1, \dots, t_{k+1})$  instead of  $\underline{t}$ , i.e.

$$\begin{aligned} D_{k,(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) &= G_{k,(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) \\ &+ H_{k,(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) + F_{k,(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}). \end{aligned} \quad (2.11)$$

If (2.11) is shown, then by using induction arguments and the relation (2.10) for  $S \in \{D, G, H, F\}$  we obtain (2.8) with  $(m_1, \dots, m_i)$  in place of  $\underline{m}$ , and  $(t_1, \dots, t_i)$  in place of  $\underline{t}$  for  $i \in \{k+2, k+3, \dots, d\}$ . In particular, for  $i = d$  we have formula (2.8).

Thus, it remains to show (2.11). By the definitions of  $G_{k,(r,s)}^{(\underline{m})}(\underline{t})$  and  $F_{k,(r,s)}^{(\underline{m})}(\underline{t})$  with  $(m_1, \dots, m_{k+1})$  instead of  $\underline{m}$  and  $(t_1, \dots, t_{k+1})$  instead of  $\underline{t}$ , we have

$$G_{k,(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) = \frac{D_{k,(r,s)}^{\sharp, (m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) - D_{k,(r,s)}^{b, (m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1})}{e^{it_k} - 1}$$

and

$$F_{k,(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) = F_{k,(r,s)}^{\sharp, (m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) - F_{k,(r,s)}^{b, (m_1, \dots, m_k)}(t_1, \dots, t_{k+1}).$$



Therefore, (2.11) is equivalent to

$$\begin{aligned} D_{(r,s)}^{(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) + \frac{1}{e^{it_k} - 1} D_{k,(r,s)}^{b,(m_1, \dots, m_{k+1})}(\dots) + F_{k,(r,s)}^{b,(m_1, \dots, m_{k+1})}(\dots) \\ = \frac{e^{it_k}}{e^{it_k} - 1} D_{k,(r,s)}^{\sharp,(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) + F_{k,(r,s)}^{\sharp,(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}). \end{aligned} \quad (2.12)$$

Now, we observe that for  $S \in \{D, D^{\sharp}, F^{\sharp}\}$  the equation in (2.10) is satisfied also for  $i = k + 1$ . Hence, (2.9) implies that (2.12) and, therefore, (2.11) is equivalent to

$$\begin{aligned} \frac{1}{e^{it_k} - 1} \sum_{\gamma_{k+1} = \lceil rm_{k+1} \rceil}^{\lfloor sm_{k+1} \rfloor} e^{i\gamma_{k+1}t_{k+1}} e^{i\lceil \gamma_{k+1}m_k/m_{k+1} \rceil t_k} D_{k,(\gamma_{k+1}/m_{k+1}, s)}^{o,(m_1, \dots, m_k)}(t_1, \dots, t_k) \\ = \frac{1}{e^{it_k} - 1} D_{k,(r,s)}^{b,(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}) + F_{k,(r,s)}^{b,(m_1, \dots, m_{k+1})}(t_1, \dots, t_{k+1}). \end{aligned} \quad (2.13)$$

But (2.13) easily follows from

$$e^{i\gamma_{k+1}t_{k+1}} e^{i\lceil \gamma_{k+1}m_k/m_{k+1} \rceil t_k} = e^{i\gamma_{k+1}(t_{k+1} + t_k m_k/m_{k+1})} e^{i\lceil \gamma_{k+1}m_k/m_{k+1} \rceil t_k}. \quad (2.14)$$

Thus, we get (2.12) and therefore (2.11).

Finally, we consider the case  $k = 1$ . Equation (2.14) yields

$$D_{(r,s)}^{(m_1, m_2)} = G_{1,(r,s)}^{(m_1, m_2)} + F_{1,(r,s)}^{(m_1, m_2)} = G_{1,(r,s)}^{(m_1, m_2)} + H_{1,(r,s)}^{(m_1, m_2)} + F_{1,(r,s)}^{(m_1, m_2)}.$$

Thus, induction arguments and the relation (2.10) for  $S \in \{D, G, H, F\}$  yield that for  $i \in \{3, 4, \dots, d\}$  we have (2.8) with  $(m_1, \dots, m_i)$  in place of  $\underline{m}$ , and  $(t_1, \dots, t_i)$  in place of  $\underline{t}$ . In particular, for  $i = d$  we have the assertion (2.8).  $\square$

**Proposition 2.6** *Let  $r, s \in \mathbb{R}$ ,  $r < s$ , be fixed. Then, for all  $\underline{m} \in [1, \infty)^d$  and all  $k \in \{1, \dots, d\}$  we have*

$$\|G_{k,(r,s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_k + 1) \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi]^{d-1})}.$$

*Proof.* By using the inequality

$$\frac{1}{|e^{it} - 1|} \lesssim \frac{1}{|t|}, \quad t \in [-\pi, \pi) \setminus \{0\}, \quad (2.15)$$

it is easy to see that for all  $m_1, m_d \in [1, \infty)$  we have

$$\int_{[-\pi, \pi)} \left| \frac{e^{i(\lfloor rm_1 \rfloor + 1)t_1} - 1}{e^{it_1} - 1} \right| dt_1 \lesssim \ln(m_1 + 1) \quad (2.16)$$

and

$$\int_{[-\pi, \pi)} \left| \frac{e^{i\lceil rm_d \rceil t_d} - 1}{e^{it_d} - 1} \right| dt_d \lesssim \ln(m_d + 1). \quad (2.17)$$

Let  $k \in \{2, \dots, d\}$ . Denoting

$$\underline{A}_k(m_k) = \left\{ \underline{t} \in [-\pi, \pi)^d \mid |t_k| \leq \frac{1}{m_k + 1} \right\}, \quad \underline{B}_k(m_k) = [-\pi, \pi)^d \setminus \underline{A}_k(m_k), \quad (2.18)$$

we have

$$\int_{[-\pi, \pi)^d} \left| \frac{\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})}{e^{it_k} - 1} \right| d\underline{t} = \int_{\underline{A}_k(m_k)} \left| \frac{\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})}{e^{it_k} - 1} \right| d\underline{t} + \int_{\underline{B}_k(m_k)} \left| \frac{\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})}{e^{it_k} - 1} \right| d\underline{t} = I + J. \quad (2.19)$$

By using (2.3) and (2.15), for all  $\underline{m} \in [1, \infty)^d$ , we obtain

$$\begin{aligned} J &\leq \int_{\underline{B}_k(m_k)} \frac{|D_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})| + |D_{k,(r,s)}^{\circ,(\underline{m})}(\underline{t})|}{|e^{it_k} - 1|} d\underline{t} \lesssim \int_{\underline{B}_k(m_k)} \frac{1}{|t_k|} |D_{k,(r,s)}^{\circ,(\underline{m})}(\underline{t})| d\underline{t} \\ &\lesssim \ln(m_k + 1) \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_k)}\|_{L^1([-\pi, \pi)^{d-1})} \end{aligned} \quad (2.20)$$

and

$$I \lesssim \int_{\underline{A}_k(m_k)} \frac{1}{|t_k|} |\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})| d\underline{t}. \quad (2.21)$$

In the following, we will use the next two well-known statements:

For all continuously differentiable  $2\pi$ -periodic  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $\delta \in \mathbb{R}$  (see [11, p. 46]):

$$\|g(\cdot + \delta) - g\|_{L^1([-\pi, \pi])} \leq |\delta| \|g'\|_{L^1([-\pi, \pi])}. \quad (2.22)$$

For all trigonometric polynomials  $\tau_n$  of degree at most  $n$ , one has (see [11, p. 102]):

$$\|\tau_n'\|_{L^1([-\pi, \pi])} \leq n \|\tau_n\|_{L^1([-\pi, \pi])}. \quad (2.23)$$

Denoting  $D_{(r,s)}^{\circ} = D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_k)}$  and  $\delta = t_k m_k / m_{k-1}$ , we can write

$$\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t}) = D_{(r,s)}^{\circ}(t_1, \dots, t_{k-2}, t_{k-1} + \delta, t_{k+1}, \dots, t_d) - D_{(r,s)}^{\circ}(t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_d).$$

Since the degree of the trigonometric polynomial  $D_{(r,s)}^{\circ}(t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_d)$  in the variable  $t_{k-1}$  is at most  $\lfloor s m_{k-1} \rfloor$ , using (2.22) and (2.23), we obtain

$$\int_{[-\pi, \pi]} |\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})| dt_{k-1} \leq |t_k| \frac{m_k}{m_{k-1}} s m_{k-1} \int_{[-\pi, \pi]} |D_{(r,s)}^{\circ}(t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_k)| dt_{k-1}.$$

Thus, since (2.21), we have for all  $\underline{m} \in [1, \infty)^d$  that

$$I \lesssim m_k \int_{-1/(m_k+1)}^{1/(m_k+1)} dt_k \|D_{(r,s)}^{\circ}\|_{L^1([-\pi, \pi)^{d-1})} \lesssim \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_k)}\|_{L^1([-\pi, \pi)^{d-1})}.$$

Combining this with (2.20), (2.19) yields: For  $\underline{m} \in [1, \infty)^d$ ,  $k \in \{2, \dots, d\}$ , we have

$$\int_{[-\pi, \pi]^d} \left| \frac{\Delta_{k,(r,s)}^{\sharp,(\underline{m})}(\underline{t})}{e^{it_k} - 1} \right| d\underline{t} \lesssim \ln(m_k + 1) \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_k)}\|_{L^1([-\pi, \pi)^{d-1})}. \quad (2.24)$$

In analogy to (2.24), we derive that for all  $\underline{m} \in [1, \infty)^d$  and all  $k \in \{1, \dots, d-1\}$

$$\int_{[-\pi, \pi]^d} \left| \frac{\Delta_{k,(r,s)}^{\flat,(\underline{m})}(\underline{t})}{e^{it_k} - 1} \right| d\underline{t} \lesssim \ln(m_k + 1) \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_k)}\|_{L^1([-\pi, \pi)^{d-1})}. \quad (2.25)$$

Finally, having in mind the definition of  $G_{k,(r,s)}^{(\underline{m})}$ , we finish the proof by combining the inequalities (2.24), (2.25), (2.16), and (2.17).  $\square$

**Proposition 2.7** *Let  $r, s \in \mathbb{R}$ ,  $r < s$ , be fixed. Then,*

a) *for all  $\underline{m} \in \mathbb{N}^d$  and all  $k \in \{2, \dots, d\}$ , we have*

$$\|F_{k,(r,s)}^{\sharp,(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_{k-1} + 1) \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi)^{d-1})}, \quad (2.26)$$

b) *for all  $\underline{m} \in \mathbb{N}^d$  and all  $k \in \{1, \dots, d-1\}$ , we have*

$$\|F_{k,(r,s)}^{\flat,(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_{k+1} + 1) \|D_{(r,s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi)^{d-1})}. \quad (2.27)$$

*Proof.* Let  $\mathbf{k} \in \{2, \dots, \mathbf{d}\}$ . We will show (2.26) for all  $\underline{\mathbf{m}} \in \mathbb{N}^{\mathbf{d}}$ . Denote

$$\begin{aligned} Q_{\mathbf{k},\nu}^{(\underline{\mathbf{m}})}(t_1, \dots, t_{\mathbf{k}-1}, t_{\mathbf{k}+1}, \dots, t_{\mathbf{d}}) \\ = \sum^{\text{ok}} e^{\mathbf{i}(\gamma_1 t_1 + \dots + \gamma_{\mathbf{k}-1} t_{\mathbf{k}-1} + \gamma_{\mathbf{k}+1} t_{\mathbf{k}+1} + \dots + \gamma_{\mathbf{d}} t_{\mathbf{d}})} \llbracket \gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m_{\mathbf{k}-1} \rrbracket^{\nu}, \end{aligned} \quad (2.28)$$

where  $\sum^{\text{ok}}$  is given by (2.4). Using the equality

$$\frac{1}{t_{\mathbf{k}}} \left( e^{-\mathbf{i} \llbracket \gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m_{\mathbf{k}-1} \rrbracket t_{\mathbf{k}}} - 1 \right) = \sum_{\nu=1}^{\infty} \frac{1}{\nu!} (-\mathbf{i})^{\nu} \llbracket \gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m_{\mathbf{k}-1} \rrbracket^{\nu} t_{\mathbf{k}}^{\nu-1},$$

and (2.15), we obtain that for all  $\underline{\mathbf{m}} \in \mathbb{N}^{\mathbf{d}}$  and all  $\underline{\mathbf{t}} \in [-\pi, \pi)^{\mathbf{d}}$

$$\left| F_{\mathbf{k},(r,s)}^{\sharp,(\underline{\mathbf{m}})}(\underline{\mathbf{t}}) \right| \lesssim \sum_{\nu=1}^{\infty} \frac{\pi^{\nu-1}}{\nu!} \left| Q_{\mathbf{k},\nu}^{(\underline{\mathbf{m}})}(t_1, \dots, t_{\mathbf{k}-2}, t_{\mathbf{k}-1} + t_{\mathbf{k}} m_{\mathbf{k}} / m_{\mathbf{k}-1}, t_{\mathbf{k}+1}, \dots, t_{\mathbf{d}}) \right|.$$

We conclude that for all  $\underline{\mathbf{m}} \in \mathbb{N}^{\mathbf{d}}$  the following inequality holds

$$\|F_{\mathbf{k},(r,s)}^{\sharp,(\underline{\mathbf{m}})}\|_{L^1([-\pi, \pi)^{\mathbf{d}})} \lesssim \sum_{\nu=1}^{\infty} \frac{\pi^{\nu}}{\nu!} \|Q_{\mathbf{k},\nu}^{(\underline{\mathbf{m}})}\|_{L^1([-\pi, \pi)^{\mathbf{d}-1})}.$$

Thus, to prove (2.26) it is sufficient to verify that for all  $\nu \geq 1$  and all  $\underline{\mathbf{m}} \in \mathbb{N}^{\mathbf{d}}$

$$\|Q_{\mathbf{k},\nu}^{(\underline{\mathbf{m}})}\|_{L^1([-\pi, \pi)^{\mathbf{d}-1})} \lesssim \ln(m_{\mathbf{k}-1} \nu + 1) \|D_{(r,s)}^{(m_1, \dots, m_{\mathbf{k}-1}, m_{\mathbf{k}+1}, \dots, m_{\mathbf{d}})}\|_{L^1([-\pi, \pi)^{\mathbf{d}-1})}. \quad (2.29)$$

For this we will use the 1-periodic function  $h_{\nu,m}$ ,  $m \geq 1$ , determined by

$$h_{\nu,m}(t) = \begin{cases} t^{\nu} & \text{if } 0 \leq t \leq 1 - m^{-1}, \\ m(1 - m^{-1})^{\nu} (1 - t) & \text{if } 1 - m^{-1} \leq t < 1. \end{cases} \quad (2.30)$$

Let us abbreviate  $m = m_{\mathbf{k}-1}$ . Since  $\gamma_{\mathbf{k}-1}$ ,  $m$ , and  $m_{\mathbf{k}}$  are integers, we have that  $0 \leq \llbracket \gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m \rrbracket \leq 1 - m^{-1}$ . Thus, taking into account that by 1-periodicity of  $h_{\nu,m}$  we have  $h_{\nu,m}(t) = h_{\nu,m}(\llbracket t \rrbracket)$ ,  $t \in \mathbb{R}$ , we derive

$$h_{\nu,m}(\gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m) = \llbracket \gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m \rrbracket^{\nu}. \quad (2.31)$$

Next, by the Fourier inversion theorem, it holds

$$h_{\nu,m}(t) = \sum_{\mu \in \mathbb{Z}} \hat{h}_{\nu,m}(\mu) e^{2\pi \mathbf{i} \mu t} \quad \text{in } L^1([0, 1]), \quad (2.32)$$

where

$$\hat{h}_{\nu,m}(\nu) = \int_{[0,1)} h_{\nu,m}(t) e^{-2\pi \mathbf{i} \nu t} dt.$$

Combining (2.28), (2.32), (2.31), we get that  $Q_{\mathbf{k},\nu}^{(\underline{\mathbf{m}})}(t_1, \dots, t_{\mathbf{k}-1}, t_{\mathbf{k}+1}, \dots, t_{\mathbf{d}})$  equals

$$\begin{aligned} \sum^{\text{ok}} e^{\mathbf{i}(\gamma_1 t_1 + \dots + \gamma_{\mathbf{k}-1} t_{\mathbf{k}-1} + \gamma_{\mathbf{k}+1} t_{\mathbf{k}+1} + \dots + \gamma_{\mathbf{d}} t_{\mathbf{d}})} \sum_{\mu \in \mathbb{Z}} \hat{h}_{\nu,m}(\mu) e^{2\pi \mathbf{i} \mu \gamma_{\mathbf{k}-1} m_{\mathbf{k}} / m} \\ = \sum_{\mu \in \mathbb{Z}} \hat{h}_{\nu,m}(\mu) D_{(r,s)}^{(m_1, \dots, m_{\mathbf{k}-1}, m_{\mathbf{k}+1}, \dots, m_{\mathbf{d}})}(t_1, \dots, t_{\mathbf{k}-2}, t_{\mathbf{k}-1} + 2\pi \mu m_{\mathbf{k}} / m, t_{\mathbf{k}+1}, \dots, t_{\mathbf{d}}) \end{aligned}$$

in  $L^1([-\pi, \pi)^{\mathbf{d}-1})$ . Hence, we have for all  $\underline{\mathbf{m}} \in \mathbb{N}^{\mathbf{d}}$  that

$$\|Q_{\mathbf{k},\nu}^{(\underline{\mathbf{m}})}\|_{L^1([-\pi, \pi)^{\mathbf{d}-1})} \lesssim \|D_{(r,s)}^{(m_1, \dots, m_{\mathbf{k}-1}, m_{\mathbf{k}+1}, \dots, m_{\mathbf{d}})}\|_{L^1([-\pi, \pi)^{\mathbf{d}-1})} \sum_{\mu \in \mathbb{Z}} \left| \hat{h}_{\nu,m}(\mu) \right|. \quad (2.33)$$

In [27] (see also [2]), it is shown that for all  $\nu, m \geq 1$ , we have

$$\sum_{\mu \in \mathbb{Z}} |\hat{h}_{\nu, m}(\mu)| \lesssim \ln(m\nu + 1). \quad (2.34)$$

Combining this inequality and (2.33), we get (2.29) and, therefore, we have (2.26).

By analogy, we can prove (2.27).  $\square$

*Proof of Theorem 2.2.* Inequality (2.1) is well-known for  $d = 1$ , since

$$L(\Xi_{(r, s)}) = \|D_{(r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)}. \quad (2.35)$$

Let  $d \geq 2$ . For all  $k \in \{1, \dots, d\}$ , we get by Proposition 2.5 that

$$\begin{aligned} \|D_{(r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} &\leq \|G_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} + \|H_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \\ &\quad + \|F_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)}. \end{aligned} \quad (2.36)$$

Clearly, for all  $\underline{m} \in \mathbb{N}^d$  and all  $k \in \{1, \dots, d\}$ , we have

$$\|H_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \leq \|D_{(r, s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi]^{d-1})} \quad (2.37)$$

and, by Proposition 2.6, for all  $\underline{m} \in \mathbb{N}^d$  and all  $k \in \{1, \dots, d\}$  we have

$$\|G_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_k + 1) \|D_{(r, s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi]^{d-1})}. \quad (2.38)$$

Thus, we need to estimate only  $\|F_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)}$ . This is done with a particular choice of the index  $k$ . Let  $k = k(\underline{m})$  be such that  $m_i \leq m_k$  for all  $i \in \{1, \dots, d\}$ . We consider the following three cases:

(i) If  $k = k(\underline{m}) \in \{2, \dots, d-1\}$ , we have

$$\|F_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \leq \|F_{k, (r, s)}^{\sharp, (\underline{m})}\|_{L^1([-\pi, \pi]^d)} + \|F_{k, (r, s)}^b(\underline{m})\|_{L^1([-\pi, \pi]^d)}$$

and

$$\ln(m_{k-1} + 1) + \ln(m_{k+1} + 1) \leq 2 \ln(m_k + 1).$$

(ii) If  $k = k(\underline{m}) = 1$ , we have

$$\|F_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} = \|F_{k, (r, s)}^b(\underline{m})\|_{L^1([-\pi, \pi]^d)} \quad \text{and} \quad \ln(m_{k+1} + 1) \leq \ln(m_k + 1).$$

(iii) If  $k = k(\underline{m}) = d$ , we have

$$\|F_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} = \|F_{k, (r, s)}^{\sharp, (\underline{m})}\|_{L^1([-\pi, \pi]^d)} \quad \text{and} \quad \ln(m_{k-1} + 1) \leq \ln(m_k + 1).$$

Therefore, by Proposition 2.7, we get that for all  $\underline{m} \in \mathbb{N}^d$  and  $k = k(\underline{m})$  we have

$$\|F_{k, (r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_k + 1) \|D_{(r, s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi]^{d-1})}. \quad (2.39)$$

Combining (2.36), (2.37), (2.38) and (2.39), we get that for all  $\underline{m} \in \mathbb{N}^d$  and  $k = k(\underline{m})$

$$\|D_{(r, s)}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_k + 1) \|D_{(r, s)}^{(m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_d)}\|_{L^1([-\pi, \pi]^{d-1})}.$$

Because of (2.35), we get the assertion (2.1) by a simple induction argument.  $\square$

## 2.2 Proof of Theorem 2.1

Let  $r, s \in \mathbb{R}$ ,  $s \geq r \geq 0$ ,  $\underline{m} = (m_1, \dots, m_d) \in (0, \infty)^d$ ,  $d \in \mathbb{N}$ . Let  $S_d$  be the set of all permutations of  $\{1, \dots, d\}$ , i.e. the set of bijections from  $\{1, \dots, d\}$  onto  $\{1, \dots, d\}$ .

For  $\sigma \in S_d$  and  $(\triangleleft_0, \dots, \triangleleft_d) \in \{\leq, <, =\}^{d+1}$ , let

$$\Xi_{(r,s),\sigma,(\triangleleft_0,\dots,\triangleleft_d)}^{(\underline{m})} = \left\{ \gamma \in \mathbb{Z}^d \mid r \triangleleft_d \frac{\gamma_{\sigma(d)}}{m_{\sigma(d)}} \triangleleft_{d-1} \dots \triangleleft_1 \frac{\gamma_{\sigma(1)}}{m_{\sigma(1)}} \triangleleft_0 s \right\}. \quad (2.40)$$

**Proposition 2.8** *Let  $r, s \in \mathbb{R}$ ,  $s \geq r \geq 0$ , be fixed. Then, for all  $\underline{m} \in \mathbb{N}^d$ , all  $\sigma \in S_d$ , and all  $(\triangleleft_0, \dots, \triangleleft_d) \in \{\leq, <, =\}^{d+1}$ , we have*

$$L\left(\Xi_{(r,s),\sigma,(\triangleleft_0,\dots,\triangleleft_d)}^{(\underline{m})}\right) \lesssim \prod_{i=1}^d \ln(m_i + 1). \quad (2.41)$$

*Proof.* Note that if (2.41) is proved for the identity permutation  $\sigma = \text{id}|_{\{1,\dots,d\}}$ , then (2.41) immediately follows for all  $\sigma \in S_d$ . Furthermore, we can restrict the considerations to  $(\triangleleft_0, \dots, \triangleleft_d) \in \{\leq, <\}^{d+1}$ . Thus, the proof follows the lines of the proof of Theorem 2.2 in an obvious way.  $\square$

We will use sets of the form (2.40) as building blocks in order to prove the upper estimate in Theorem 2.1. Let us formulate a technical auxiliary statement.

**Lemma 2.9** *Let  $\mathfrak{X}^{(\underline{m})}$  be a set of subsets of  $\mathbb{R}^d$  and  $\underline{m} \in \mathbb{N}^d$ . For  $N \in \mathbb{N}$  we denote*

$$\begin{aligned} \mathfrak{X}_{\cap,N}^{(\underline{m})} &= \left\{ \Xi_1 \cap \dots \cap \Xi_j \mid \Xi_1, \dots, \Xi_j \in \mathfrak{X}^{(\underline{m})}, j \in \{1, \dots, N\} \right\}, \\ \mathfrak{X}_{\cup,N}^{(\underline{m})} &= \left\{ \Xi_1 \cup \dots \cup \Xi_j \mid \Xi_1, \dots, \Xi_j \in \mathfrak{X}^{(\underline{m})}, j \in \{1, \dots, N\} \right\}. \end{aligned} \quad (2.42)$$

*Assume that  $N \in \mathbb{N}$  is fixed and that for  $\underline{m} \in \mathbb{N}^d$  and all  $\Xi \in \mathfrak{X}_{\cap,N}^{(\underline{m})}$  we have*

$$L(\Xi) \lesssim \prod_{i=1}^d \ln(m_i + 1). \quad (2.43)$$

*Then, for the fixed  $N \in \mathbb{N}$ , the estimate (2.43) holds also for all  $\Xi \in \mathfrak{X}_{\cup,N}^{(\underline{m})}$ .*

*Proof.* The well-known inclusion–exclusion principle yields

$$L(\Xi_1 \cup \dots \cup \Xi_j) \leq \sum_{k=1}^j \left( \sum_{1 \leq l_1 < \dots < l_k \leq j} L(\Xi_{l_1} \cap \dots \cap \Xi_{l_k}) \right).$$

Since  $N$  is fixed, we conclude the assertion.  $\square$

For  $\underline{m} \in \mathbb{N}^d$ , we consider the sets

$$\overline{\Gamma}_0^{(\underline{m})} = \left\{ \gamma \in \mathbb{N}_0^d \mid \forall i: 2\gamma_i \leq m_i \right\}, \quad \overline{\Gamma}_1^{(\underline{m})} = \left\{ \gamma \in \mathbb{N}_0^d \mid \forall i: 2\gamma_i < m_i \right\},$$

and we use the notation

$$K^{(\underline{m})}[\gamma] = \{ i \in \{1, \dots, d\} \mid \gamma_i/m_i = \max^{(\underline{m})}[\gamma] \} \quad (2.44)$$

with  $\max^{(\underline{m})}[\gamma] = \max\{\gamma_i/m_i \mid i \in \{1, \dots, d\}\}$ , and for  $\emptyset \neq K \subseteq \{1, \dots, d\}$ , we denote

$$\overline{\Gamma}_1^{(\underline{m}),K} = \left\{ \gamma \in \overline{\Gamma}_1^{(\underline{m})} \mid K^{(\underline{m})}[\gamma] = K \right\}.$$

**Proposition 2.10** *Let  $d \geq 2$  and  $\emptyset \neq K = \{k_1, \dots, k_h\} \subsetneq \{1, \dots, d\}$ ,  $k_1 < \dots < k_h$ . Then  $\overline{\Gamma}_1^{(\underline{m}),K}$  is equal to*

$$\bigcup_{\sigma \in S_{d,K}} \left\{ \gamma \in \mathbb{N}_0^d \mid 0 \leq \frac{\gamma_{\sigma(d)}}{m_{\sigma(d)}} \leq \dots \leq \frac{\gamma_{\sigma(h+1)}}{m_{\sigma(h)}} < \frac{\gamma_{\sigma(h)}}{m_{\sigma(h)}} = \dots = \frac{\gamma_{\sigma(1)}}{m_{\sigma(1)}} < \frac{1}{2} \right\}, \quad (2.45)$$

where  $S_{d,K} = \{ \sigma \in S_d \mid \sigma(1) = k_1, \dots, \sigma(h) = k_h \}$ .

*Proof.* By the definition, we have

$$\overline{\Gamma}_1^{(\underline{m}),K} = \left\{ \gamma \in \mathbb{N}_0^d \mid \forall j \notin K : 0 \leq \frac{\gamma_j}{m_j} < \frac{\gamma_{k_h}}{m_{k_h}} = \dots = \frac{\gamma_{k_1}}{m_{k_1}} < \frac{1}{2} \right\}. \quad (2.46)$$

Since for  $\sigma \in S_{d,K}$  we have  $\sigma(1) = k_1, \dots, \sigma(h) = k_h$  and  $\sigma(h+1), \dots, \sigma(d) \notin K$ , we conclude that (2.45) is a subset of (2.46).

Now, let  $\gamma$  be an element of (2.46). Then, there exist  $j_{h+1}, \dots, j_d$  such that

$$\{j_{h+1}, \dots, j_d\} = \{1, \dots, d\} \setminus K \quad \text{and} \quad \frac{\gamma_{j_d}}{m_{j_d}} \leq \dots \leq \frac{\gamma_{j_{h+1}}}{m_{j_{h+1}}}.$$

We set  $\sigma(1) = k_1, \dots, \sigma(h) = k_h$  and  $\sigma(h+1) = j_{h+1}, \dots, \sigma(d) = j_d$ . Then,  $\sigma \in S_{d,K}$  and  $\gamma$  is an element of the corresponding set in the union (2.45).  $\square$

**Corollary 2.11** *Let  $\emptyset \neq K \subseteq \{1, \dots, d\}$ . Then, for all  $\underline{m} \in \mathbb{N}^d$ , we have*

$$L(\overline{\Gamma}_1^{(\underline{m}),K}) \lesssim \prod_{i=1}^d \ln(m_i + 1). \quad (2.47)$$

*Proof.* For  $K = \{1, \dots, d\}$ , we have

$$\overline{\Gamma}_1^{(\underline{m}),\{1,\dots,d\}} = \left\{ \gamma \in \mathbb{N}_0^d \mid 0 \leq \frac{\gamma_d}{m_d} = \dots = \frac{\gamma_1}{m_1} < \frac{1}{2} \right\} = \Xi_{(0,1/2),\text{id},(\leq,=,\dots,=,<)}^{(\underline{m})}.$$

Thus, Proposition 2.8 implies (2.47).

Let us now consider the case  $d \geq 2$  with a non-empty set  $\emptyset \neq K \subsetneq \{1, \dots, d\}$ . Let  $h \in \{1, \dots, d\}$  be the same as in Proposition 2.10. Let  $\triangleleft_0, \triangleleft_h$  be the relation  $<$ . If  $h \geq 2$ , let further  $(\triangleleft_1, \dots, \triangleleft_{h-1}) = (=, \dots, =)$  and  $(\triangleleft_{h+1}, \dots, \triangleleft_d) = (\leq, \dots, \leq)$ . With this notation Proposition 2.10 implies that

$$\overline{\Gamma}_1^{(\underline{m}),K} = \bigcup_{\sigma \in S_{d,K}} \Xi_{(0,1/2),\sigma,(\triangleleft_0,\dots,\triangleleft_d)}^{(\underline{m})}. \quad (2.48)$$

Let  $\mathfrak{X}^{(\underline{m})} = \left\{ \Xi_{(0,1/2),\sigma,(\triangleleft_0,\dots,\triangleleft_d)}^{(\underline{m})} \mid \sigma \in S_{d,K} \right\}$  and  $N = (d-h)!$ . Then, (2.42) equals

$$\mathfrak{X}_{\cap,N}^{(\underline{m})} = \left\{ \Xi_{(0,1/2),\sigma,(\triangleleft'_0,\dots,\triangleleft'_d)}^{(\underline{m})} \mid \sigma \in S_{d,K}, \triangleleft'_j \in \{\leq, =\} \text{ if } h+1 \leq j \leq d-1, \triangleleft'_j = \triangleleft_j \text{ else} \right\}.$$

Since Proposition 2.8 implies (2.43) for all sets in  $\mathfrak{X}_{\cap,N}^{(\underline{m})}$ , Lemma 2.9 yields (2.43) for all sets in  $\mathfrak{X}_{\cup,N}^{(\underline{m})}$ . Now, taking into account that by (2.48) we have  $\overline{\Gamma}_1^{(\underline{m}),K} \in \mathfrak{X}_{\cup,N}^{(\underline{m})}$ , we obtain the assertion (2.47).  $\square$

For  $k \in \{1, \dots, d\}$ , we define

$$\mathfrak{s}_k^{(\underline{m})}(\gamma) = (\gamma_1, \dots, \gamma_{k-1}, m_k - \gamma_k, \gamma_{k-1}, \dots, \gamma_d). \quad (2.49)$$

**Proposition 2.12** *For  $\underline{m} \in \mathbb{N}^d$ , we have*

$$\overline{\Gamma}^{(\underline{m})} = \overline{\Gamma}_0^{(\underline{m})} \cup \bigcup_{\emptyset \neq K \subseteq \{1,\dots,d\}} \bigcup_{k \in K} \mathfrak{s}_k^{(\underline{m})}(\overline{\Gamma}_1^{(\underline{m}),K}) \quad (2.50)$$

*and, furthermore, the right hand side of (2.50) is a union of pairwise disjoint sets.*

*Proof.* Let  $\gamma \in \overline{\Gamma}^{(\underline{m})}$ . We will show that  $\gamma$  belongs to the right hand side of (2.50). Since this is clear if  $\gamma \in \overline{\Gamma}_0^{(\underline{m})}$ , we assume that  $\gamma \notin \overline{\Gamma}_0^{(\underline{m})}$ . Since  $\gamma \notin \overline{\Gamma}_0^{(\underline{m})}$ , there exists

$k$  such that  $\gamma_k/m_k > 1/2$ . Therefore, by the definition of  $\bar{\Gamma}^{(\underline{m})}$ , we have

$$\forall i \in \{1, \dots, d\} \setminus \{k\} : \quad \gamma_i/m_i < 1/2. \quad (2.51)$$

Let  $\gamma' = \mathfrak{s}_k^{(\underline{m})}(\gamma)$ . Since  $\gamma_k/m_k > 1/2$ , we have  $\gamma'_k/m_k < 1/2$ . Thus, since for  $i \neq k$  we have  $\gamma'_i = \gamma_i$ , we get from (2.51) that  $\gamma' \in \bar{\Gamma}_1^{(\underline{m})}$ . By the definition of  $\bar{\Gamma}^{(\underline{m})}$ , we have

$$\forall i \in \{1, \dots, d\} \setminus \{k\} : \quad \gamma'_i/m_i - \gamma'_k/m_k = \gamma_i/m_i + \gamma_k/m_k - 1 \leq 1 - 1 = 0.$$

Thus, by the definition in (2.44), we have  $k \in K^{(\underline{m})}[\gamma']$ . Obviously  $\gamma' = \mathfrak{s}_k^{(\underline{m})}(\gamma)$  implies  $\gamma = \mathfrak{s}_k^{(\underline{m})}(\gamma')$ . Thus, we have  $\gamma \in \mathfrak{s}_k^{(\underline{m})}(\bar{\Gamma}_1^{(\underline{m}),K})$  with  $K = K^{(\underline{m})}[\gamma']$  and  $k \in K$ .

Now, let  $\gamma$  belong to the right hand side of (2.50). We will show that  $\gamma \in \bar{\Gamma}^{(\underline{m})}$ . This is clear if  $\gamma \in \bar{\Gamma}_0^{(\underline{m})}$ . Suppose  $\gamma \in \mathfrak{s}_k^{(\underline{m})}(\bar{\Gamma}_1^{(\underline{m}),K})$  with  $K \subseteq \{1, \dots, d\}$  and  $k \in K$ . There is  $\gamma' \in \bar{\Gamma}_1^{(\underline{m}),K}$  with  $\gamma = \mathfrak{s}_k^{(\underline{m})}(\gamma')$  and, by the definition of  $\bar{\Gamma}_1^{(\underline{m}),K}$ , we have  $K = K^{(\underline{m})}[\gamma']$ . Since  $k \in K = K^{(\underline{m})}[\gamma']$ , we have  $\gamma'_i/m_i \leq \gamma'_k/m_k$ ,  $i \in \{1, \dots, d\}$ , thus

$$\forall i \in \{1, \dots, d\} : \quad \gamma_i/m_i + \gamma_k/m_k = \gamma'_i/m_i - \gamma'_k/m_k + 1 \leq 1. \quad (2.52)$$

Since for  $j \neq k$  we have  $\gamma_j = \gamma'_j$ , and since  $\gamma' \in \bar{\Gamma}_1^{(\underline{m})}$ , we have (2.51), and therefore

$$\forall i, j \in \{1, \dots, d\} \setminus \{k\} : \quad \gamma_i/m_i + \gamma_j/m_j < 1. \quad (2.53)$$

Combining (2.52) and (2.53) yields  $\gamma \in \bar{\Gamma}^{(\underline{m})}$ .

Finally, to complete the proof, we show that the right hand side of (2.50) is the union of pairwise disjoint sets. Let  $\gamma \in \mathfrak{s}_k^{(\underline{m})}(\bar{\Gamma}_1^{(\underline{m}),K})$  and  $k \in K$ . Then,  $\gamma_k > m_k/2$  and therefore  $\gamma \notin \bar{\Gamma}_0^{(\underline{m})}$ . Let furthermore  $\gamma \in \mathfrak{s}_{k'}^{(\underline{m})}(\bar{\Gamma}_1^{(\underline{m}),K'})$ . Then,  $\gamma_{k'} > m_{k'}/2$ . Therefore, since  $\gamma \in \bar{\Gamma}^{(\underline{m})}$ , we have  $k' = k$ , for otherwise  $1 < \gamma_k/m_k + \gamma_{k'}/m_{k'} \leq 1$ . We have  $\gamma = \mathfrak{s}_k^{(\underline{m})}(\gamma')$  for some  $\gamma'$  that is uniquely determined by  $\gamma' = \mathfrak{s}_k^{(\underline{m})}(\gamma)$ . Therefore,  $\gamma' \in \bar{\Gamma}_1^{(\underline{m}),K}$  and  $\gamma' \in \bar{\Gamma}_1^{(\underline{m}),K'}$ , and we conclude  $K' = K^{(\underline{m})}[\gamma'] = K$ .  $\square$

**Corollary 2.13** *For all  $\underline{m} \in \mathbb{N}^d$ , we have*

$$L(\bar{\Gamma}^{(\underline{m})}) \lesssim \prod_{i=1}^d \ln(m_i + 1).$$

*Proof.* By Proposition 2.12, the right hand side of (2.50) is a union of pairwise disjoint sets. Therefore, (2.50) implies

$$L(\bar{\Gamma}^{(\underline{m})}) \leq L(\bar{\Gamma}_0^{(\underline{m})}) + \sum_{\emptyset \neq K \subseteq \{1, \dots, d\}} \sum_{k \in K} L(\mathfrak{s}_k^{(\underline{m})}(\bar{\Gamma}_1^{(\underline{m}),K})). \quad (2.54)$$

Clearly

$$L(\mathfrak{s}_k^{(\underline{m})}(\bar{\Gamma}_1^{(\underline{m}),K})) = L(\bar{\Gamma}_1^{(\underline{m}),K}) \quad (2.55)$$

and the cross product structure of  $\bar{\Gamma}_0^{(\underline{m})}$  implies

$$L(\bar{\Gamma}_0^{(\underline{m})}) \lesssim \prod_{i=1}^d \ln(m_i + 1). \quad (2.56)$$

Combining (2.54), (2.55), Corollary 2.11, and (2.56) yields the assertion.  $\square$

**Corollary 2.14** For all  $\underline{m} \in \mathbb{N}^d$ , we have

$$L(\underline{\Gamma}^{(\underline{m}),*}) \lesssim \prod_{i=1}^d \ln(m_i + 1).$$

*Proof.* For  $\underline{u} \in \{-1, 1\}^d$ , we denote  $\underline{\Gamma}_{\underline{u}}^{(\underline{m})} = \{\gamma \in \mathbb{Z}^d \mid (u_1 \gamma_1, \dots, u_d \gamma_d) \in \underline{\Gamma}^{(\underline{m})}\}$ . Consider  $\mathfrak{X}^{(\underline{m})} = \{\underline{\Gamma}_{\underline{u}}^{(\underline{m})} \mid \underline{u} \in \{-1, 1\}^d\}$  and  $N = 2^d$ . Then, it is clear that

$$\underline{\Gamma}^{(\underline{m}),*} = \bigcup_{\underline{u} \in \{-1, 1\}^d} \underline{\Gamma}_{\underline{u}}^{(\underline{m})} \in \mathfrak{X}_{\cup, N}^{(\underline{m})}. \quad (2.57)$$

Let  $\underline{u}^{(1)}, \dots, \underline{u}^{(j)} \in \{-1, 1\}^d$ , and  $M = \{i \in \{1, \dots, d\} \mid u_i^{(1)} = u_i^{(2)} = \dots = u_i^{(j)}\}$ . We have  $\bigcap_{l=1}^j \underline{\Gamma}_{\underline{u}^{(l)}}^{(\underline{m})} = \{\gamma \in \underline{\Gamma}_{\underline{u}^{(1)}}^{(\underline{m})} \mid \gamma_i = 0 \text{ for all } i \notin M\}$ . If  $\emptyset \neq M = \{i_1, \dots, i_h\}$ ,  $i_1 < \dots < i_h$ ,  $(m'_1, \dots, m'_h) = (m_{i_1}, \dots, m_{i_h})$ ,  $(u'_1, \dots, u'_h) = (u_{i_1}^{(1)}, \dots, u_{i_h}^{(1)})$ , then

$$L\left(\bigcap_{l=1}^j \underline{\Gamma}_{\underline{u}^{(l)}}^{(\underline{m})}\right) = L\left(\underline{\Gamma}_{(u'_1, \dots, u'_h)}^{(m'_1, \dots, m'_h)}\right) = L\left(\underline{\Gamma}^{(m'_1, \dots, m'_h)}\right). \quad (2.58)$$

At the same time, if  $M = \emptyset$ , then the left hand side in (2.58) is  $L(\{\underline{0}\}) = 1$ .

Note that  $\prod_{l=1}^h \ln(m'_l + 1) \leq \prod_{i=1}^d \ln(m_i + 1)$  for  $M \neq \emptyset$ . Thus, using Corollary 2.13 we conclude that for all  $\underline{m} \in \mathbb{N}^d$  we have

$$L\left(\bigcap_{l=1}^j \underline{\Gamma}_{\underline{u}^{(l)}}^{(\underline{m})}\right) \lesssim \prod_{i=1}^d \ln(m_i + 1). \quad (2.59)$$

Now, (2.59) implies that the assumption (2.43) is satisfied and, therefore, taking into account (2.57) and Lemma 2.9 we get the assertion.  $\square$

**Lemma 2.15** For  $z > 0$  and  $a \in (0, 1]$ , we have

$$\max\{\ln(az), 1\} \geq \max\{a \ln z, 1\} \geq a \max\{\ln z, 1\} \geq a \ln z. \quad (2.60)$$

*Proof.* The assertion is trivial for  $a = 1$ . Let  $a \in (0, 1)$ . The function  $h : (0, 1] \rightarrow \mathbb{R}$ ,  $h(u) = u(1 - \ln u)$ ,  $u \in (0, 1]$ , is increasing, thus  $a(1 - \ln a) < h(1) = 1$ .

We conclude  $\ln a > (a - 1)(1 - \ln a)$ , i.e.  $\ln a > (a - 1) \ln(e/a)$ . Thus, since  $a - 1 < 0$ , for  $z \geq e/a$  we have  $\ln a > (a - 1) \ln z$ , i.e.  $\ln(az) > a \ln z$ . For  $0 < z < e/a$ , we conclude  $a \ln z < a \ln(e/a) = a(1 - \ln a) < 1$ . We have shown: if  $\ln(az) \geq 1$ , then we have  $\ln(az) > a \ln z$ , and if  $\ln(az) < 1$ , then we have also  $a \ln z < 1$ .  $\square$

**Proposition 2.16** There exists  $\alpha_d, \beta_d > 0$  such that for all  $\underline{m} \in \mathbb{N}^d$ , we have

$$L(\underline{\Gamma}^{(\underline{m})}) \geq \alpha_d \prod_{i=1}^d \ln(m_i + 1), \quad L(\underline{\Gamma}^{(\underline{m}),*}) \geq \beta_d \prod_{i=1}^d \ln(m_i + 1). \quad (2.61)$$

*Proof.* We use the following Hardy-Littlewood inequality, see [29, p. 286]:

$$\frac{1}{2} \int_{[0, 2\pi)} \left| \sum_{\gamma=0}^N c_\gamma e^{i\gamma t} \right| dt \geq \sum_{\gamma=0}^N \frac{|c_\gamma|}{\gamma + 1}, \quad N \in \mathbb{N}_0, \quad c_0, \dots, c_N \in \mathbb{C}. \quad (2.62)$$



By the induction argument from [21, p. 69], we get for  $N_1, \dots, N_d \in \mathbb{N}_0$ ,  $c_{\underline{\gamma}} \in \mathbb{C}^d$ :

$$\frac{1}{2^d} \int_{[0, 2\pi]^d} \left| \sum_{\gamma_1=0}^{N_1} \cdots \sum_{\gamma_d=0}^{N_d} c_{\underline{\gamma}} e^{i(\underline{\gamma}, \underline{t})} \right| d\underline{t} \geq \sum_{\gamma_1=0}^{N_1} \cdots \sum_{\gamma_d=0}^{N_d} \frac{|c_{\underline{\gamma}}|}{(\gamma_1 + 1) \cdots (\gamma_d + 1)}. \quad (2.63)$$

Using an appropriate shifting and orthogonality, we obtain

$$L(\underline{\Gamma}) \geq \frac{1}{\pi^d} \quad \text{for all finite } \emptyset \neq \underline{\Gamma} \subset \mathbb{Z}^d. \quad (2.64)$$

By (2.63), we get for  $L(\underline{\Gamma}^{(m)})$  the lower bounds

$$\frac{1}{\pi^d} \sum_{\underline{\gamma} \in \underline{\Gamma}^{(m)}} \frac{1}{(\gamma_1 + 1) \cdots (\gamma_d + 1)} \geq \frac{1}{\pi^d} \sum_{\gamma_1=0}^{\lfloor m_1/2 \rfloor} \cdots \sum_{\gamma_d=0}^{\lfloor m_d/2 \rfloor} \frac{1}{(\gamma_1 + 1) \cdots (\gamma_d + 1)}.$$

Since

$$\sum_{\gamma=0}^{\lfloor x \rfloor} \frac{1}{\gamma + 1} \geq \max\{\ln(x + 1), 1\}, \quad x \geq 0, \quad (2.65)$$

we have

$$L(\underline{\Gamma}^{(m)}) \geq \frac{1}{\pi^d} \prod_{i=1}^d \max\{\ln(m_i/2 + 1), 1\} \geq \frac{1}{\pi^d} \prod_{i=1}^d \max\{\ln((m_i + 1)/2), 1\},$$

and now Lemma 2.15 implies the first inequality in (2.61) with  $\alpha_d = (2\pi)^{-d}$ .

Since  $L(\{\gamma \in \mathbb{Z} \mid |\gamma| \leq x\}) = L(\{\gamma \in \mathbb{Z} \mid 0 \leq \gamma \leq 2\lfloor x \rfloor\})$ ,  $x \geq 0$ , the Hardy-Littlewood inequality (2.62) and (2.65) imply

$$L(\{\gamma \in \mathbb{Z} \mid |\gamma| \leq x\}) \geq \frac{1}{\pi} \sum_{\gamma=0}^{2\lfloor x \rfloor} \frac{1}{\gamma + 1} \geq \frac{1}{\pi} \max(\ln(x + 1), 1), \quad x \geq 0. \quad (2.66)$$

Thus, for  $d = 1$ , (2.66) yields the second inequality in (2.61) with  $\beta_1 = \pi^{-1}$ . Let us prove this inequality for  $d \geq 2$ . We adapt the decomposition approach from [28]. For  $j \in \{1, \dots, d\}$ , we denote

$$\begin{aligned} \underline{\Gamma}_{j,(\gamma)}^{(m),*} &= \left\{ (\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_d) \mid (\gamma_1, \dots, \gamma_{j-1}, \gamma, \gamma_{j+1}, \dots, \gamma_d) \in \underline{\Gamma}^{(m),*} \right\}, \\ a_{j,(\gamma)}^{(m)}(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_d) &= \sum_{(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_d) \in \underline{\Gamma}_{j,(\gamma)}^{(m),*}} e^{\gamma_1 t_1 + \dots + \gamma_{j-1} t_{j-1} + \gamma_{j+1} t_{j+1} + \dots + \gamma_d t_d}. \end{aligned}$$

Using the following equality

$$\sum_{\underline{\gamma} \in \underline{\Gamma}^{(m),*}} e^{i(\underline{\gamma}, \underline{t})} = e^{-i m_j t_j} \sum_{\gamma=0}^{2m_j} e^{i \gamma t_j} a_{j,(\gamma-m_j)}^{(m)}(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_d)$$

and (2.62), we get

$$\frac{1}{2} \int_{[0, 2\pi]} \left| \sum_{\underline{\gamma} \in \underline{\Gamma}^{(m),*}} e^{i(\underline{\gamma}, \underline{t})} \right| dt_j \geq \sum_{\gamma=0}^{2m_j} \frac{1}{\gamma + 1} |a_{j,(\gamma-m_j)}^{(m)}(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_d)|$$

and, therefore, we derive

$$L(\underline{\Gamma}^{(m),*}) \geq \frac{1}{\pi} \sum_{\gamma=0}^{\lfloor m_j/2 \rfloor} \frac{1}{\gamma + 1} L(\underline{\Gamma}_{j,(\gamma-m_j)}^{(m),*}). \quad (2.67)$$

Denote  $K = \{i \in \{1, \dots, d\} \mid \ln((m_i + 1)/(4e^d)) \geq 1\}$ . Having in mind (2.64), we can assume without restriction  $K \neq \emptyset$ , since we can ensure  $\beta_d \in (0, \pi^{-d}]$ .

Let  $j \in K$ . For all  $\gamma \in \{0, \dots, \lfloor m_j/2 \rfloor\}$ , we have the cross product structure

$$\underline{\Gamma}_{j, (m_j - \gamma)}^{(\underline{m}),*} = \left\{ (\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_d) \mid \forall i \in \{1, \dots, d\} \setminus \{j\} : |\gamma_i| \leq \frac{m_i}{m_j} \gamma \right\}.$$

Thus, for all  $\gamma \in \{0, \dots, \lfloor m_j/2 \rfloor\}$  the inequality (2.66) implies

$$L(\underline{\Gamma}_{j, (m_j - \gamma)}^{(\underline{m}),*}) \geq \frac{1}{\pi^{d-1}} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} \gamma + 1 \right). \quad (2.68)$$

Note that the product over the empty set  $K \setminus \{j\} = \emptyset$  is considered as 1. In this case, by (2.64), the inequality (2.68) is satisfied. Now, (2.67) yields

$$L(\underline{\Gamma}^{(\underline{m}),*}) \geq \frac{1}{\pi^d} \sum_{\gamma=1}^{\lfloor m_j/2 \rfloor} \frac{1}{\gamma+1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{1}{2} \frac{m_i}{m_j} (2\gamma) + 1 \right).$$

For  $\gamma \geq 1$ , we have  $2\gamma \geq \gamma + 1$ . Further, we have  $4 \leq 2m_j$ . We conclude

$$\begin{aligned} \pi^d L(\underline{\Gamma}^{(\underline{m}),*}) + \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{4} + 1 \right) &\geq \frac{1}{\pi^d} \sum_{\gamma=0}^{\lfloor m_j/2 \rfloor} \frac{1}{\gamma+1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{1}{2} \frac{m_i}{m_j} (\gamma+1) + 1 \right) \\ &\geq \int_0^{m_j/2} \frac{1}{v+1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{1}{2} \frac{m_i}{m_j} v + 1 \right) dv. \end{aligned} \quad (2.69)$$

Next, for  $r > 0$ , we derive

$$\begin{aligned} \sum_{j \in K} \int_0^{rm_j} \frac{1}{v+1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{1}{2} \frac{m_i}{m_j} v + 1 \right) dv &= \sum_{j \in K} \int_0^{r/2} \frac{m_j}{m_j \tau + \frac{1}{2}} \prod_{i \in K \setminus \{j\}} \ln (m_i \tau + 1) d\tau \\ &\geq \int_0^{r/2} \sum_{j \in K} \frac{m_j}{m_j \tau + 1} \prod_{i \in K \setminus \{j\}} \ln (m_i \tau + 1) d\tau = \prod_{i \in K} \ln \left( \frac{1}{2} r m_i + 1 \right) \end{aligned}$$

and, therefore, there exists  $k \in K$  such that

$$\int_0^{rm_k} \frac{1}{v+1} \prod_{i \in K \setminus \{k\}} \ln \left( \frac{1}{2} \frac{m_i}{m_k} v + 1 \right) dv \geq \frac{1}{|K|} \prod_{i \in K} \ln \left( \frac{1}{2} r m_i + 1 \right). \quad (2.70)$$

Using (2.70) with  $r = 1/2$  and (2.69) with  $j = k$  and taking into account the definition of  $K$ , we obtain

$$\begin{aligned} \pi^d L(\underline{\Gamma}^{(\underline{m}),*}) &\geq \frac{1}{d} \left( \ln \left( \frac{1}{4} m_k + 1 \right) - \ln(e^d) \right) \prod_{i \in K \setminus \{k\}} \ln \left( \frac{1}{4} m_i + 1 \right) \\ &\geq \frac{1}{d} \prod_{i \in K} \ln((m_i + 1)/(4e^d)) = \frac{1}{d} \prod_{i=1}^d \max\{\ln((m_i + 1)/(4e^d)), 1\}. \end{aligned} \quad (2.71)$$

Now, Lemma 2.15 implies the assertion with  $\beta_d = d^{-1} \pi^{-d} (4e^d)^{-d} \in (0, \pi^{-d}]$ .  $\square$

*Proof of Theorem 2.1.* The statement follows immediately by combining Corollary 2.13, Corollary 2.14, and Proposition 2.16.  $\square$

## 2.3 Proof of Theorem 2.3

Let  $d \geq 2$ ,  $\underline{m} \in (0, \infty)^d$ , and  $r > 0$ . Denote  $D_{\Sigma, r}^{(\underline{m})}(\underline{t}) = \sum_{\underline{\gamma} \in \underline{\Sigma}_r^{(\underline{m})}} e^{i(\underline{\gamma}, \underline{t})}$  and

$$\lambda_r^{(m_1, \dots, m_j)}(\gamma_1, \dots, \gamma_{j-1}) = m_j \left( r - \sum_{i=1}^{j-1} \frac{\gamma_i}{m_i} \right), \quad j = 2, \dots, d.$$

It is easy to see that

$$D_{\Sigma,r}^{(\underline{m})}(\underline{t}) = \sum_{\gamma_1=0}^{\lfloor rm_1 \rfloor} e^{i\gamma_1 t_1} \sum_{\gamma_2=0}^{\lfloor \lambda_r^{(m_1, m_2)}(\gamma_1) \rfloor} e^{i\gamma_2 t_2} \dots \sum_{\gamma_d=0}^{\lfloor \lambda_r^{(m_1, \dots, m_d)}(\gamma_1, \dots, \gamma_{d-1}) \rfloor} e^{i\gamma_d t_d}.$$

In what follows, we will need several auxiliary functions given by

$$\begin{aligned} F_{\Sigma,r}^{(\underline{m})}(\underline{t}) &= \sum_{\gamma_1=0}^{\lfloor rm_1 \rfloor} e^{i\gamma_1(t_1 - m_d t_d/m_1)} \sum_{\gamma_2=0}^{\lfloor \lambda_r^{(m_1, m_2)}(\gamma_1) \rfloor} e^{i\gamma_2(t_2 - m_d t_d/m_2)} \dots \\ &\dots \sum_{\gamma_{d-1}=0}^{\lfloor \lambda_r^{(m_1, \dots, m_{d-1})}(\gamma_1, \dots, \gamma_{d-2}) \rfloor} e^{i\gamma_{d-1}(t_{d-1} - m_d t_d/m_{d-1})} f_{\Sigma,r}^{(\underline{m})}(\gamma_1, \dots, \gamma_{d-1}, t_d), \end{aligned} \quad (2.72)$$

$$f_{\Sigma,r}^{(\underline{m})}(\gamma_1, \dots, \gamma_{d-1}, t_d) = e^{i(rm_d+1)t_d} \frac{e^{-i\lfloor \lambda_r^{(m_1, \dots, m_d)}(\gamma_1, \dots, \gamma_{d-1}) \rfloor t_d} - 1}{e^{it_d} - 1}, \quad (2.73)$$

and

$$\begin{aligned} D_{\Sigma,d,r}^{\circ,(\underline{m})}(\underline{t}) &= D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}(t_1, \dots, t_{d-1}), \\ D_{\Sigma,d,r}^{(\underline{m})}(\underline{t}) &= D_{\Sigma,r}^{(\underline{m})}(t_1 - m_d t_d/m_1, \dots, t_{d-1} - m_d t_d/m_{d-1}), \\ G_{\Sigma,r}^{(\underline{m})}(\underline{t}) &= \frac{1}{e^{it_d} - 1} \left( e^{i(rm_d+1)t_d} D_{\Sigma,d,r}^{(\underline{m})}(\underline{t}) - D_{\Sigma,d,r}^{\circ,(\underline{m})}(\underline{t}) \right). \end{aligned} \quad (2.74)$$

**Proposition 2.17** *We have  $D_{\Sigma,r}^{(\underline{m})}(\underline{t}) = G_{\Sigma,r}^{(\underline{m})}(\underline{t}) + F_{\Sigma,r}^{(\underline{m})}(\underline{t})$ .*

*Proof.* We have for  $i \in \{1, \dots, d-2\}$  the recursive relation

$$S_{\Sigma,r}^{(m_i, \dots, m_d)}(t_i, \dots, t_d) = \sum_{\gamma_i=0}^{\lfloor rm_i \rfloor} e^{i\gamma_i t_i} S_{\Sigma, (r-\gamma_i/m_i)}^{(m_{i+1}, \dots, m_d)}(t_{i+1}, \dots, t_d), \quad (2.75)$$

where  $S \in \{D, F\}$ . Note that

$$\begin{aligned} G_{\Sigma,r}^{(m_{d-1}, m_d)}(t_{d-1}, t_d) &= \frac{1}{e^{it_d} - 1} \left( e^{i(rm_d+1)t_d} D_r^{(m_{d-1})}(t_{d-1} - m_d t_d/m_{d-1}) - D_r^{(m_{d-1})}(t_{d-1}) \right), \\ F_{\Sigma,r}^{(m_{d-1}, m_d)}(t_{d-1}, t_d) &= \frac{e^{i(m_d+1)t_d}}{e^{it_d} - 1} \sum_{\gamma_{d-1}=0}^{\lfloor rm_{d-1} \rfloor} e^{i\gamma_{d-1}(t_{d-1} - m_d t_d/m_{d-1})} \left( e^{-i\lfloor m_d(r-\gamma_{d-1}/m_{d-1}) \rfloor t_d} - 1 \right). \end{aligned}$$

Thus, from (2.75) for  $S = D$ , we immediately get the same recursive relation (2.75) for the function corresponding to the symbol  $S = G$ .

Next, using the equality

$$e^{i\gamma_{d-1} t_{d-1}} e^{i(\lfloor m_d(r-\gamma_{d-1}/m_{d-1}) \rfloor + 1)t_d} = e^{i(rm_d+1)t_d} e^{i\gamma_{d-1}(t_{d-1} - m_d t_d/m_{d-1})} e^{-i\lfloor m_d(r-\gamma_{d-1}/m_{d-1}) \rfloor t_d},$$

we conclude that  $D_{\Sigma,r}^{(m_{d-1}, m_d)} = G_{\Sigma,r}^{(m_{d-1}, m_d)} + F_{\Sigma,r}^{(m_{d-1}, m_d)}$ . Thus, applying the relations (2.75) to  $S \in \{D, G, F\}$ , we obtain the assertion.  $\square$

**Proposition 2.18** *Let  $r \in (0, \infty)$ . For all  $\underline{m} \in [1, \infty)^d$ , we have*

$$\|G_{\Sigma,r}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_d + 1) \|D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}\|_{L^1([-\pi, \pi]^{d-1})}.$$

*Proof.* We have

$$G_{\Sigma,r}^{(\underline{m})}(\underline{t}) = \frac{\Delta_{\Sigma,r}^{(\underline{m})}(\underline{t})}{e^{it_d} - 1} + L_r^{(m_d)}(t_d) D_r^{(m_1, \dots, m_{d-1})}(t_1 - m_d t_d/m_1, \dots, t_{d-1} - m_d t_d/m_{d-1}),$$

where  $L_r^{(m)}(t) = \frac{e^{i(rm+1)t} - 1}{e^{it} - 1}$  and  $\Delta_{\Sigma,r}^{(\underline{m})}(\underline{t}) = D_{\Sigma,d,r}^{(\underline{m})}(\underline{t}) - D_{\Sigma,d,r}^{\circ,(\underline{m})}(\underline{t})$ .

Moreover, by the telescoping sum decomposition, we derive

$$\Delta_{\Sigma,r}^{(\underline{m})}(\underline{t}) = \sum_{i=1}^{d-1} \Delta_{\Sigma,r,i}^{(\underline{m})}(\underline{t}), \quad \text{where} \quad (2.76)$$

$$\begin{aligned} \Delta_{\Sigma,r,i}^{(\underline{m})}(\underline{t}) = & D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}(t_1, \dots, t_{i-1}, t_i - m_d t_d / m_i, \dots, t_{d-1} - m_d t_d / m_{d-1}) \\ & - D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}(t_1, \dots, t_{i-1}, t_i, t_{i+1} - m_d t_d / m_{i+1}, \dots, t_{d-1} - m_d t_d / m_{d-1}). \end{aligned}$$

Using (2.76), (2.74), and the sets (2.18), we get

$$\int_{[-\pi, \pi)} |G_{\Sigma,r}^{(\underline{m})}| d\underline{t} \leq \sum_{i=1}^d I_i + J,$$

where

$$I_i = \int_{\underline{A}_d(m_d)} \frac{|\Delta_{\Sigma,r,i}^{(\underline{m})}(\underline{t})|}{|e^{it_d} - 1|} d\underline{t} \lesssim \int_{\underline{A}_d(m_d)} \frac{1}{|t_d|} |\Delta_{\Sigma,r,i}^{(\underline{m})}(\underline{t})| d\underline{t},$$

and

$$J = \int_{\underline{B}_d(m_d)} \frac{|D_{\Sigma,d,r}^{(\underline{m})}(\underline{t})| + |D_{\Sigma,d,r}^{\circ,(\underline{m})}(\underline{t})|}{|e^{it_d} - 1|} d\underline{t} + \int_{\underline{A}_d(m_d)} |L_r^{m_d}(t_d) D_{\Sigma,d,r}^{(\underline{m})}(\underline{t})| d\underline{t}.$$

By (2.15), we easily get  $J \lesssim \ln(m_d + 1) \|D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}\|_{L^1([-\pi, \pi)^{d-1})}$ . Further, we have

$$I_i \lesssim m_d \int_{-1/(m_d+1)}^{1/(m_d+1)} dt_d \|D_{(r,s)}\|_{L^1([-\pi, \pi)^{d-1})} \lesssim \|D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}\|_{L^1([-\pi, \pi)^{d-1})}. \quad (2.77)$$

Indeed, using (2.22) and (2.23), we obtain for  $i \in \{1, \dots, d-1\}$  that

$$\int_{[-\pi, \pi)} |\Delta_{\Sigma,r,i}^{(\underline{m})}(\underline{t})| dt_i \leq |t_i| \frac{m_d}{m_i} r m_i \int_{[-\pi, \pi)} |D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}(t_1, \dots, t_{d-1})| dt_i,$$

and, therefore (2.77).  $\square$

**Proposition 2.19** *Let  $r = p/q$  with  $p, q \in \mathbb{N}$ . For all  $\underline{m} \in \mathbb{N}^d$ , we have*

$$\|F_{\Sigma,r}^{(\underline{m})}\|_{L^1([-\pi, \pi)^d)} \lesssim \ln(\text{lcm}(q, m_1, \dots, m_{d-1}) + 1) \|D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}\|_{L^1([-\pi, \pi)^{d-1})},$$

where  $\text{lcm}(q, m_1, \dots, m_{d-1})$  denotes the least common multiple of  $q, m_1, \dots, m_{d-1}$ .

*Proof.* The proposition can be proved by repeating the proof of Proposition 2.7. Thus, let us present the sketch of the proof.

Using (2.72), (2.73), and (2.15), we get as in the proof of Proposition 2.7 that

$$\|F_{\Sigma,r}^{(\underline{m})}\|_{L^1([-\pi, \pi)^d)} \lesssim \sum_{\nu=1}^{\infty} \frac{\pi^\nu}{\nu!} \|Q_{\Sigma,\nu}^{(\underline{m})}\|_{L^1([-\pi, \pi)^{d-1})},$$

where

$$Q_{\Sigma,\nu}^{(\underline{m})}(t_1, \dots, t_{d-1}) = \sum_{\gamma_1=0}^{\lfloor r m_1 \rfloor} \sum_{\gamma_2=0}^{\lfloor \lambda(\gamma_1) \rfloor} \dots \sum_{\gamma_{d-1}=0}^{\lfloor \lambda(\gamma_1, \dots, \gamma_{d-2}) \rfloor} e^{i(t_1 \gamma_1 + \dots + t_{d-1} \gamma_{d-1})} \|\lambda(\gamma_1, \dots, \gamma_{d-1})\|^\nu$$

and  $\lambda(\gamma_1, \dots, \gamma_{j-1}) = \lambda_r^{(m_1, \dots, m_j)}(\gamma_1, \dots, \gamma_{j-1})$ .

Thus, to finish the proof it is sufficient to verify that for all  $\nu \geq 1$  we have

$$\|Q_{\Sigma,\nu}^{(\underline{m})}\|_{L^1([-\pi, \pi)^{d-1})} \lesssim \ln(M\nu + 1) \|D_{\Sigma,r}^{(m_1, \dots, m_{d-1})}\|_{L^1([-\pi, \pi)^{d-1})}, \quad (2.78)$$

where  $M = \text{lcm}(q, m_1, \dots, m_{d-1})$ .

Taking into account that  $0 \leq \|\lambda(\gamma_1, \dots, \gamma_{d-1})\| \leq 1 - M^{-1}$ , we get in the same way as in the proof of Proposition 2.7 that in  $L^1([-\pi, \pi]^{d-1})$

$$\begin{aligned} & Q_{\Sigma, \nu}^{(\underline{m})}(t_1, \dots, t_{d-1}) \\ &= \sum_{\gamma_1=0}^{\lfloor rm_1 \rfloor} \dots \sum_{\gamma_{d-1}=0}^{\lfloor \lambda(\gamma_1, \dots, \gamma_{d-2}) \rfloor} e^{i(t_1 \gamma_1 + \dots + t_{d-1} \gamma_{d-1})} \sum_{\mu \in \mathbb{Z}} \hat{h}_{\nu, M}(\mu) e^{2\pi i \mu \lambda(\gamma_1, \dots, \gamma_{d-1})} \\ &= \sum_{\mu \in \mathbb{Z}} \hat{h}_{\nu, M}(\mu) e^{2\pi i \mu m_d} D_{\Sigma, r}^{(m_1, \dots, m_{d-1})}(t_1 - 2\pi \mu m_d / m_1, \dots, t_{d-1} - 2\pi \mu m_d / m_{d-1}), \end{aligned}$$

where the function  $h_{\nu, M}$  is given by (2.30). Thus, using (2.34), we get (2.78).  $\square$

*Proof of Theorem 2.3.* The statement of the theorem is well-known for  $d = 1$ . Remark also that the case  $d = 2$  with  $r = 1$  is already considered in Theorem 2.1.

Let us prove the upper estimates for  $d \geq 2$  in (2.2). Without loss of generality we can assume that  $m_1 \leq \dots \leq m_d$ . The upper estimate for  $L(\underline{\Sigma}_r^{(\underline{m})}) = \|D_{\Sigma, r}^{(\underline{m})}\|_{L^1([-\pi, \pi]^d)}$  can now be easily obtained by using Proposition 2.17, Proposition 2.18, Proposition 2.19 and the induction argument. Using this, we can conclude the upper estimate for  $L(\underline{\Sigma}_r^{(\underline{m}),*})$  in the same way as in the proof of Corollary 2.14.

Let us consider the lower bounds. As in the proof of Proposition 2.16, we get

$$\begin{aligned} L(\underline{\Sigma}_r^{(\underline{m})}) &\geq \frac{1}{\pi^d} \sum_{\gamma_1=0}^{\lfloor rm_1/d \rfloor} \dots \sum_{\gamma_d=0}^{\lfloor rm_d/d \rfloor} \frac{1}{(\gamma_1 + 1) \dots (\gamma_d + 1)} \\ &\geq \frac{1}{\pi^d} (\min\{r/d, 1\})^d \prod_{i=1}^d \ln(m_i + 1). \end{aligned}$$

To show the lower bounds for the sets  $\underline{\Sigma}_r^{(\underline{m}),*}$ , it is sufficient to prove that there exists  $\kappa_d \in (0, \pi^{-d}]$  such that for all  $r > 0$  and all  $\underline{m} \in \mathbb{N}^d$ , we have

$$L(\underline{\Sigma}_r^{(\underline{m}),*}) \geq \kappa_d \prod_{i=1}^d \max\{\ln(rm_i + 1), 1\}, \quad (2.79)$$

since by Lemma 2.15 we will have  $L(\underline{\Sigma}_r^{(\underline{m}),*}) \geq \kappa_d (\min\{r, 1\})^d \prod_{i=1}^d \ln(m_i + 1)$ .

We use the induction argument. By (2.66), we can choose  $\kappa_1 = \pi^{-1}$ . Let  $d \geq 2$ . By analogy with the proof of the second inequality in (2.61), we get

$$L(\underline{\Sigma}_r^{(\underline{m}),*}) \geq \frac{1}{\pi} \sum_{\gamma=0}^{\lfloor rm_j \rfloor} \frac{1}{\gamma + 1} L(\underline{\Sigma}_{r, j, (\lfloor rm_j \rfloor - \gamma)}^{(\underline{m}),*}), \quad (2.80)$$

where  $\underline{\Sigma}_{r, j, (\gamma)}^{(\underline{m}),*} = \{(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_d) \mid (\gamma_1, \dots, \gamma_{j-1}, \gamma, \gamma_{j+1}, \dots, \gamma_d) \in \underline{\Sigma}_r^{(\underline{m}),*}\}$ . Denote  $K = \{i \in \{1, \dots, d\} \mid \ln((rm_i + 1)/(4e^d)) \geq 1\}$ . We can assume without restriction that  $K \neq \emptyset$ , since by (2.64) the number  $\kappa_d$  can be chosen from  $(0, \pi^{-d}]$ .

Let  $j \in K$ . For  $\gamma \in \{0, \dots, \lfloor rm_j \rfloor\}$ , we have

$$\underline{\Sigma}_{r, j, (\lfloor rm_j \rfloor - \gamma)}^{(\underline{m}),*} = \underline{\Sigma}_{r-r/m_j+\gamma/m_j}^{(m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_d),*}.$$

Thus, since  $r - r/m_j + \gamma/m_j \geq \gamma/m_j$ , using the induction argument yields

$$L(\underline{\Sigma}_{r, j, (\lfloor rm_j \rfloor - \gamma)}^{(\underline{m}),*}) \geq \kappa_{d-1} \prod_{\substack{i=1 \\ i \neq j}}^d \max\{\ln((\gamma/m_j)m_i + 1), 1\} \geq \kappa_{d-1} \prod_{i \in K \setminus \{j\}} \ln((\gamma/m_j)m_i + 1).$$

Note again that the product over the empty set  $K \setminus \{j\} = \emptyset$  is considered as 1, and that in this case by (2.64) the last inequality is satisfied with  $\kappa_{d-1} \in (0, \pi^{-(d-1)}]$ .

We have  $r \geq 1/m_j$ . By analogy with (2.69), using (2.80) implies

$$\begin{aligned} L\left(\underline{\Sigma}_r^{(\underline{m}),*}\right) + \frac{\kappa_{d-1}}{\pi} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{1}{2} r m_i + 1 \right) \\ \geq \frac{\kappa_{d-1}}{\pi} \int_0^{r m_j} \frac{1}{v+1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{1}{2} \frac{m_i}{m_j} v + 1 \right) dv. \end{aligned}$$

There is  $k \in K$  satisfying (2.70). Using the first and second inequality in (2.60), by analogy with (2.71), we get (2.79) for  $\kappa_d = d^{-1} \pi^{-1} (4e^d)^{-d} \kappa_{d-1} \in (0, \pi^{-d}]$ .  $\square$

### 3 Interpolation on Lissajous-Chebyshev nodes

We first describe the solution of the interpolation problem (1.2) in more detail and collect some notation from [9].

Let us consider for  $\underline{\gamma} \in \mathbb{N}_0^d$  the  $d$ -variate Chebyshev polynomials

$$T_{\underline{\gamma}}(\underline{x}) = T_{\gamma_1}(x_1) \cdot \dots \cdot T_{\gamma_d}(x_d), \quad \underline{x} \in [-1, 1]^d,$$

where  $T_\gamma(x) = \cos(\gamma \arccos x)$ . The Chebyshev polynomials form an orthogonal basis of the polynomial space  $\Pi^d = \text{span}\{T_{\underline{\gamma}} \mid \underline{\gamma} \in \mathbb{N}_0^d\}$  with respect to the inner product

$$\langle f, g \rangle_{w_d} = \frac{1}{\pi^d} \int_{[-1, 1]^d} f(\underline{x}) \overline{g(\underline{x})} w_d(\underline{x}) d\underline{x}, \quad w_d(\underline{x}) = \prod_{i=1}^d \frac{1}{\sqrt{1-x_i^2}}.$$

The corresponding norms of these basis elements are

$$\|T_{\underline{\gamma}}\|_{w_d, 2}^2 = 2^{-\epsilon(\underline{\gamma})}, \quad \text{where } \epsilon(\underline{\gamma}) = \#\{i \in \{1, \dots, d\} \mid \gamma_i \neq 0\}.$$

We define

$$\mathcal{N}_d = \{\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d \mid n_1, \dots, n_d \text{ are pairwise relatively prime}\}.$$

In the following, let  $\underline{n} \in \mathcal{N}_d$ ,  $\epsilon \in \{1, 2\}$ , and  $\underline{\kappa} \in \mathbb{Z}^d$ . For the index set  $\underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$  from the introduction, we define the polynomial vector space

$$\Pi_{\underline{\kappa}}^{(\epsilon \underline{n})} = \text{span} \left\{ T_{\underline{\gamma}} \mid \underline{\gamma} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})} \right\}.$$

Clearly, the system  $\{T_{\underline{\gamma}} \mid \underline{\gamma} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}\}$  forms an orthogonal basis of  $\Pi_{\underline{\kappa}}^{(\epsilon \underline{n})}$ .

To introduce the sets  $\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$ , we define

$$\underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})} = \underline{\mathbf{I}}_{\underline{\kappa}, 0}^{(\epsilon \underline{n})} \cup \underline{\mathbf{I}}_{\underline{\kappa}, 1}^{(\epsilon \underline{n})},$$

where the sets  $\underline{\mathbf{I}}_{\underline{\kappa}, \tau}^{(\epsilon \underline{n})}$ ,  $\tau \in \{0, 1\}$ , are given by

$$\underline{\mathbf{I}}_{\underline{\kappa}, \tau}^{(\epsilon \underline{n})} = \left\{ \underline{i} \in \mathbb{N}_0^d \mid 0 \leq i_i \leq \epsilon n_i \text{ and } i_i \equiv \kappa_i + \tau \pmod{2} \text{ for all } i \in \{1, \dots, d\} \right\}.$$

Then, using the notation

$$\underline{z}_{\underline{i}}^{(\epsilon \underline{n})} = \left( z_{i_1}^{(\epsilon n_1)}, \dots, z_{i_d}^{(\epsilon n_d)} \right), \quad z_i^{(\epsilon n)} = \cos(i\pi/(\epsilon n)),$$

the Lissajous-Chebyshev node sets are defined as

$$\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \underline{n})} = \left\{ \underline{z}_{\underline{i}}^{(\epsilon \underline{n})} \mid \underline{i} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})} \right\}. \quad (3.1)$$

Note that the mapping  $\underline{i} \mapsto \underline{z}_{\underline{i}}^{(\epsilon \underline{n})}$  is a bijection from  $\underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$  onto  $\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$ .

Further, for  $\underline{i} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$ , we introduce the weight  $\mathfrak{w}_{\underline{i}}^{(\epsilon \underline{n})}$  by

$$\mathfrak{w}_{\underline{i}}^{(\epsilon \underline{n})} = 2^{\#\{i \mid 0 < i_i < \epsilon n_i\}} / \left( 2 \epsilon^d \prod_{i=1}^d n_i \right), \quad (3.2)$$

and for  $\underline{\gamma} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$ , we use the notation

$$\mathfrak{f}^{(\epsilon \underline{n})}(\underline{\gamma}) = \max \left\{ \#\{i \in \{1, \dots, d\} \mid 2\gamma_i = \epsilon n_i\} - 1, 0 \right\}.$$

Note that in the case  $\epsilon = 1$ , we have  $\mathfrak{f}^{(\underline{n})}(\underline{\gamma}) = 0$  for all  $\underline{\gamma} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\underline{n})}$ .

Finally, for  $\underline{i} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}$ , we introduce on  $[-1, 1]^d$  the polynomials

$$L_{\underline{\kappa}, \underline{i}}^{(\epsilon \underline{n})}(\underline{x}) = \mathfrak{w}_{\underline{i}}^{(\epsilon \underline{n})} \left( \sum_{\underline{\gamma} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}} 2^{\mathfrak{f}^{(\epsilon \underline{n})}(\underline{\gamma})} T_{\underline{\gamma}}(\underline{z}_{\underline{i}}^{(\epsilon \underline{n})}) T_{\underline{\gamma}}(\underline{x}) - T_{\epsilon n_d}(z_{i_d}^{(\epsilon n_d)}) T_{\epsilon n_d}(x_d) \right) \quad (3.3)$$

that by definition belong to the space  $\Pi_{\underline{\kappa}}^{(\epsilon \underline{n})}$ .

The existence and uniqueness of a solution of the interpolation problem (1.2) are guaranteed by the following theorem.

**Theorem 3.1** *For  $f : [-1, 1]^d \rightarrow \mathbb{R}$  the unique solution to the interpolation problem (1.2) in the space  $\Pi_{\underline{\kappa}}^{(\epsilon \underline{n})}$  is given by the polynomial*

$$P_{\underline{\kappa}}^{(\epsilon \underline{n})} f(\underline{x}) = \sum_{\underline{i} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}} f(\underline{z}_{\underline{i}}^{(\epsilon \underline{n})}) L_{\underline{\kappa}, \underline{i}}^{(\epsilon \underline{n})}(\underline{x}).$$

The proof of this result is given in [9]. Note that for  $\epsilon = 1$  only the case  $\underline{\kappa} = \underline{0}$  was treated. However, since the general node sets  $\underline{\mathbf{LC}}_{\underline{\kappa}}^{(\underline{n})}$  differ from  $\underline{\mathbf{LC}}_{\underline{0}}^{(\underline{n})}$  only in terms of reflections with respect to the coordinate axis, the corresponding results can be transferred immediately.

By Theorem 3.1, the discrete Lebesgue constant  $\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})}$  introduced in (1.3) can be reformulated as

$$\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})} = \max_{\underline{x} \in [-1, 1]^d} \sum_{\underline{i} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}} |L_{\underline{\kappa}, \underline{i}}^{(\epsilon \underline{n})}(\underline{x})|. \quad (3.4)$$

As a first auxiliary result to estimate this constant, we proof the following Marcinkiewicz-Zygmund-type inequality.

**Proposition 3.2** *Let  $\underline{\kappa} \in \mathbb{Z}^d$ ,  $\epsilon \in \{1, 2\}$ ,  $\underline{n} \in \mathcal{N}_d$ , and  $0 < p < \infty$  be fixed. For all  $P \in \Pi_{\underline{\kappa}}^{(\epsilon \underline{n})}$ , we have*

$$\sum_{\underline{i} \in \underline{\mathbf{I}}_{\underline{\kappa}}^{(\epsilon \underline{n})}} \mathfrak{w}_{\underline{i}}^{(\epsilon \underline{n})} |P(\underline{z}_{\underline{i}}^{(\epsilon \underline{n})})|^p \lesssim \|P\|_{w_d, p}^p = \frac{1}{\pi^d} \int_{[-1, 1]^d} |P(\underline{x})|^p w_d(\underline{x}) d\underline{x}. \quad (3.5)$$

*Proof.* The proof is based on the idea given in [26]. We proceed as in the proof of [12, Lemma 3] and use the following one-dimensional result from [18, Theorem 2]: For all  $M \in \mathbb{N}$ ,  $0 \leq \theta_1 < \dots < \theta_M < 2\pi$ , and for all univariate trigonometric polynomials  $q_m$  of degree at most  $m \in \mathbb{N}$ , we have the inequality

$$\sum_{\nu=1}^M |q_m(\theta_\nu)|^p \leq \left( m + \frac{1}{2\eta} \right) \frac{(p+1)e}{2\pi} \int_0^{2\pi} |q_m(\theta)|^p d\theta, \quad (3.6)$$

where  $\eta = \min(\theta_2 - \theta_1, \dots, \theta_M - \theta_{M-1}, 2\pi - (\theta_M - \theta_1))$ .

For  $m \in \mathbb{N}$ ,  $\mathfrak{r} \in \{0, 1\}$ , we consider the sets  $J_{\mathfrak{r}}^{(m)} = \{i \in \mathbb{N}_0 \mid i < 2m, i \equiv \mathfrak{r} \pmod{2}\}$ . Suppose that  $i_1, \dots, i_m \in J_{\mathfrak{r}}^{(m)}$  with  $0 \leq i_1 < \dots < i_m < 2m$ . Setting  $M = m$  and  $\theta_\nu = i_\nu \pi / m$ , we obtain  $\eta = 2\pi / m$ . Using (3.6), we get for all univariate polynomials  $P$  of degree at most  $m$  the inequality

$$\begin{aligned} \frac{1}{m} \sum_{i \in J_{\mathfrak{r}}^{(m)} : i \leq m} (2 - \delta_{0,i} - \delta_{0,m}) |P(z_i^{(m)})|^p &= \frac{1}{m} \sum_{i \in J_{\mathfrak{r}}^{(m)}} |P(\cos \frac{i\pi}{m})|^p = \frac{1}{m} \sum_{\nu=1}^m |P(\cos \theta_\nu)|^p \\ &\leq \left(1 + \frac{1}{4\pi}\right) \frac{(p+1)e}{2\pi} \int_0^{2\pi} |P(\cos \theta)|^p d\theta \leq \frac{3(p+1)}{\pi} \int_{-1}^1 \frac{|P(x)|^p}{\sqrt{1-x^2}} dx, \end{aligned} \quad (3.7)$$

where  $\delta_{i,j}$  denotes the Kronecker delta. Now, taking into account the cross product structure of  $\mathbf{I}_{\underline{\kappa}, \mathfrak{r}}^{(\epsilon \mathbf{n})}$ ,  $\mathfrak{r} \in \{0, 1\}$ , the weights defined in (3.2), and applying  $\mathbf{d}$  times inequality (3.7), we obtain

$$\sum_{\mathbf{i} \in \mathbf{I}_{\underline{\kappa}, \mathfrak{r}}^{(\epsilon \mathbf{n})}} \mathfrak{w}_{\mathbf{i}}^{(\epsilon \mathbf{n})} |P(\mathbf{z}_{\mathbf{i}}^{(\epsilon \mathbf{n})})|^p \lesssim \|P\|_{w_{\mathbf{d}, p}}^p, \quad \mathfrak{r} \in \{0, 1\}, \quad (3.8)$$

for all  $P \in \Pi_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$ . Since  $\mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})} = \mathbf{I}_{\underline{\kappa}, 0}^{(\epsilon \mathbf{n})} \cup \mathbf{I}_{\underline{\kappa}, 1}^{(\epsilon \mathbf{n})}$ , inequality (3.8) yields (3.5).  $\square$

A slight adaption of the proof of Theorem 2.1 gives the following result.

**Corollary 3.3** *Let  $\epsilon \in \{1, 2\}$  and  $\underline{\kappa} \in \mathbb{Z}^{\mathbf{d}}$  be fixed. For all  $\underline{\mathbf{n}} \in \mathcal{N}_{\mathbf{d}}$ , we have*

$$\mathbf{L}(\mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}) \asymp \mathbf{L}(\mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n}), *}) \asymp \prod_{i=1}^{\mathbf{d}} \ln(n_i + 1).$$

*Proof.* Using the notation

$$\mathbf{I}_{\underline{\kappa}, \mathfrak{r}}^{(\epsilon \mathbf{n})} = \left\{ \gamma \in \mathbb{N}_0^{\mathbf{d}} \mid \begin{array}{ll} \gamma_i / n_i \leq \epsilon/2 & \forall i \in \{1, \dots, \mathbf{d}\} \text{ with } \kappa_i \equiv \mathfrak{r} \pmod{2}, \\ \gamma_i / n_i < \epsilon/2 & \forall i \in \{1, \dots, \mathbf{d}\} \text{ with } \kappa_i \not\equiv \mathfrak{r} \pmod{2} \end{array} \right\}$$

and employing the notation and statements from the proof of [9, Proposition 2.6 and Proposition 3.8], we have  $\mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})} = \mathbf{I}_{\underline{\kappa}, 0}^{(\epsilon \mathbf{n})} \cup \{\mathfrak{s}^{(\epsilon \mathbf{n})}(\gamma) \mid \gamma \in \mathbf{I}_{\underline{\kappa}, 1}^{(\epsilon \mathbf{n})}\}$ . Here, the mapping  $\mathfrak{s}^{(\epsilon \mathbf{n})}(\gamma)$  is defined as in (2.49) by  $\mathfrak{s}^{(\epsilon \mathbf{n})}(\gamma) = \mathfrak{s}_{\mathbf{k}}^{(\epsilon \mathbf{n})}(\gamma)$ , where  $\mathbf{k} = \max\{i \in \mathbf{K}^{(\mathbf{m})}[\gamma]\}$  and  $\mathbf{K}^{(\mathbf{m})}[\gamma]$  is given in (2.44). Now, if we substitute in Subsection 2.2 the symbols  $\overline{\mathbf{I}}^{(\mathbf{m})}$ ,  $\overline{\mathbf{I}}_0^{(\mathbf{m})}$ , and  $\overline{\mathbf{I}}_1^{(\mathbf{m})}$  by  $\mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}$ ,  $\mathbf{I}_{\underline{\kappa}, 0}^{(\epsilon \mathbf{n})}$  and  $\mathbf{I}_{\underline{\kappa}, 1}^{(\epsilon \mathbf{n})}$ , respectively, then the proof of the corollary follows by the same lines of argumentation as the proof of Theorem 2.1.  $\square$

We obtain the following estimates of the discrete Lebesgue constants.

**Theorem 3.4** *Let  $\epsilon \in \{1, 2\}$  and  $\underline{\kappa} \in \mathbb{Z}^{\mathbf{d}}$  be fixed. For all  $\underline{\mathbf{n}} \in \mathcal{N}_{\mathbf{d}}$ , we have*

$$\Lambda_{\underline{\kappa}}^{(\epsilon \mathbf{n})} \asymp \prod_{i=1}^{\mathbf{d}} \ln(n_i + 1). \quad (3.9)$$

*Proof.* We introduce

$$\widetilde{K}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}(\mathbf{x}, \mathbf{x}') = \sum_{\gamma \in \mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}} 2^{\epsilon(\gamma) - \mathfrak{f}^{(\epsilon \mathbf{n})}(\gamma)} T_{\gamma}(\mathbf{x}) T_{\gamma}(\mathbf{x}'), \quad K_{\underline{\kappa}}^{(\epsilon \mathbf{n})}(\mathbf{x}, \mathbf{x}') = \sum_{\gamma \in \mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}} 2^{\epsilon(\gamma)} T_{\gamma}(\mathbf{x}) T_{\gamma}(\mathbf{x}').$$

From (3.3), (3.4), and Proposition 3.2, we get for all  $\underline{\mathbf{n}} \in \mathcal{N}_{\mathbf{d}}$  that

$$\begin{aligned} \Lambda_{\underline{\kappa}}^{(\epsilon \mathbf{n})} &= \max_{\mathbf{x} \in [-1, 1]^{\mathbf{d}}} \sum_{\mathbf{i} \in \mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}} \mathfrak{w}_{\mathbf{i}}^{(\epsilon \mathbf{n})} \left| \widetilde{K}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}(\mathbf{x}, \mathbf{z}_{\mathbf{i}}^{(\epsilon \mathbf{n})}) - T_{\epsilon \mathbf{n}_{\mathbf{d}}}(z_{i_{\mathbf{d}}}^{(\epsilon \mathbf{n}_{\mathbf{d}})}) T_{\epsilon \mathbf{n}_{\mathbf{d}}}(x_{\mathbf{d}}) \right| \\ &\leq \max_{\mathbf{x} \in [-1, 1]^{\mathbf{d}}} \sum_{\mathbf{i} \in \mathbf{I}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}} \mathfrak{w}_{\mathbf{i}}^{(\epsilon \mathbf{n})} \left| \widetilde{K}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}(\mathbf{x}, \mathbf{z}_{\mathbf{i}}^{(\epsilon \mathbf{n})}) \right| + 1 \lesssim \max_{\mathbf{x} \in [-1, 1]^{\mathbf{d}}} \|\widetilde{K}_{\underline{\kappa}}^{(\epsilon \mathbf{n})}(\mathbf{x}, \cdot)\|_{w_{\mathbf{d}, 1}} + 1. \end{aligned} \quad (3.10)$$



Using the well-known relation  $\prod_{i=1}^r \cos(\vartheta_i) = \frac{1}{2^r} \sum_{\underline{v} \in \{-1,1\}^r} \cos(v_1 \vartheta_1 + \cdots + v_r \vartheta_r)$ ,  $r \in \mathbb{N}$ , we get  $\prod_{i=1}^d \cos(\gamma_i s_i) \cos(\gamma_i t_i) = \frac{1}{2^{2d}} \sum_{\underline{v}, \underline{w} \in \{-1,1\}^d} \cos\left(\sum_{i=1}^d (v_i \gamma_i s_i + w_i \gamma_i t_i)\right)$ .

Then, for all  $\underline{x} = (\cos s_1, \dots, \cos s_d)$ , we get

$$\begin{aligned} \|K_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{x}, \cdot)\|_{w_d,1} &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})} \prod_{i=1}^d \cos(\gamma_i s_i) \cos(\gamma_i t_i) \right| d\underline{t} \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \frac{1}{2^{2d}} \sum_{\underline{w} \in \{-1,1\}^d} \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})} \sum_{\underline{v} \in \{-1,1\}^d} \cos\left(\sum_{i=1}^d (v_i \gamma_i s_i + w_i v_i \gamma_i t_i)\right) \right| d\underline{t} \\ &\leq \frac{1}{2^d} \sum_{\underline{w} \in \{-1,1\}^d} \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})-d} \sum_{\underline{v} \in \{-1,1\}^d} \cos\left(\sum_{i=1}^d v_i (\gamma_i s_i + w_i \gamma_i t_i)\right) \right| d\underline{t} \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})-d} \sum_{\underline{v} \in \{-1,1\}^d} \cos\left(\sum_{i=1}^d v_i \gamma_i (s_i + t_i)\right) \right| d\underline{t} = L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}). \end{aligned} \quad (3.11)$$

Note that  $L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}) = \|K_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{1}, \cdot)\|_{w_d,1}$ . In the same way as in (3.11), we get

$$\|\widetilde{K}_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{x}, \cdot)\|_{w_d,1} \leq \|\widetilde{K}_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{1}, \cdot)\|_{w_d,1} \quad \text{for all } \underline{x} \in [-1, 1]^d.$$

Thus, by (3.10), we obtain for all  $\underline{n} \in \mathcal{N}_d$  the upper estimate

$$\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})} \lesssim \|\widetilde{K}_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{1}, \cdot)\|_{w_d,1} + 1. \quad (3.12)$$

For  $K \subseteq \{1, \dots, d\}$ , we denote  $\underline{\Gamma}_{\underline{\kappa}, K}^{(\epsilon \underline{n})} = \{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})} \mid 2\gamma_i = \epsilon n_i \Leftrightarrow i \in K\}$ . Then, for all  $\underline{x} = (\cos t_1, \dots, \cos t_d)$ , we have

$$\widetilde{K}_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{1}, \underline{x}) = K_{\underline{\kappa}}^{(\epsilon \underline{n})}(\underline{1}, \underline{x}) - \sum_{\emptyset \neq K \subseteq \{1, \dots, d\}} (1 - 2^{-\#K+1}) \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}, K}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})} \cos(\gamma_i t_i). \quad (3.13)$$

If  $K \neq \emptyset$ , then  $\underline{\Gamma}_{\underline{\kappa}, K}^{(\epsilon \underline{n})} \subseteq \underline{\Gamma}_{\underline{\kappa}, \mathbf{r}}^{(\epsilon \underline{n})}$  for some  $\mathbf{r} \in \{0, 1\}$ . Therefore, the sets  $\underline{\Gamma}_{\underline{\kappa}, K}^{(\epsilon \underline{n})}$  have a cross product structure and we get

$$\left| \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}, K}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})} \prod_{i=1}^d \cos(\gamma_i t_i) \right| \lesssim \prod_{i \in \{1, \dots, d\} \setminus K} \left| \sum_{\gamma_i = -\lfloor \epsilon n_i / 2 \rfloor}^{\lfloor \epsilon n_i / 2 \rfloor} e^{i \gamma_i t_i} \right|.$$

Thus, for  $K \neq \emptyset$  and all  $\underline{n} \in \mathcal{N}_d$ , we have

$$\int_{[0, 2\pi]^d} \left| \sum_{\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}, K}^{(\epsilon \underline{n})}} 2^{\epsilon(\underline{\gamma})} \prod_{i=1}^d \cos(\gamma_i t_i) \right| d\underline{t} \lesssim \prod_{i \in \{1, \dots, d\} \setminus K} \ln(n_i + 1). \quad (3.14)$$

Now, combining (3.11), (3.12), (3.13), and (3.14) gives the first inequality in (1.5). Finally, Corollary 3.3 implies for  $\underline{n} \in \mathcal{N}_d$  the estimate from above in (3.9).

We turn to the lower bound in (3.9). Let  $\epsilon \in \{1, 2\}$ ,  $\underline{\kappa} \in \mathbb{Z}^d$ , and  $\underline{n} \in \mathcal{N}_d$ . By [22, Theorem 1], we have

$$\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})} \geq \frac{1}{3^d} L(\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n}),*}). \quad (3.15)$$

Note that in [22] the  $L^\infty$ - $L^\infty$ -operator norm of the partial Fourier sum operator with respect to the set  $\underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}$  is used to characterize the Lebesgue constant related to Fourier sums. This characterization is identical to the definition given in this article, see [17] and the references therein. The relation (3.15) and Corollary 3.3 immediately imply that for  $\underline{n} \in \mathcal{N}_d$  we have the estimate from below in (3.9).  $\square$

To estimate the approximation error  $\|f - P_{\underline{\kappa}}^{(\epsilon \underline{n})} f\|_\infty$  for a continuous function  $f \in C([-1, 1]^d)$ , let us consider the error of the best approximation given by

$$E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f) = \min_{P \in \Pi_{\underline{\kappa}}^{(\epsilon \underline{n})}} \|f - P\|_\infty.$$

Further, let  $P^* \in \Pi_{\underline{\kappa}}^{(\epsilon \underline{n})}$  be such that  $E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f) = \|f - P^*\|_\infty$ . By Theorem 3.4, we get

$$\begin{aligned} \|f - P_{\underline{\kappa}}^{(\epsilon \underline{n})} f\|_\infty &\leq \|P_{\underline{\kappa}}^{(\epsilon \underline{n})}(P^* - f)\|_\infty + \|f - P^*\|_\infty \\ &\leq (\Lambda_{\underline{\kappa}}^{(\epsilon \underline{n})} + 1) E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f) \lesssim \left( \prod_{i=1}^d \ln(n_i + 1) \right) E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f). \end{aligned} \quad (3.16)$$

Now, using (3.16) and a multivariate version of Jackson's inequality (see [23, Section 5.3.2]) to estimate  $E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f)$ , we obtain the following result.

**Corollary 3.5** *Let  $\epsilon \in \{1, 2\}$  and  $\underline{\kappa} \in \mathbb{Z}^d$  be fixed. Let also  $\underline{s} \in \mathbb{N}_0^d$  and*

$$\frac{\partial^{s_j} f}{\partial x_j^{s_j}} \in C([-1, 1]^d), \quad j \in \{1, \dots, d\}.$$

*Then, for  $\underline{n} \in \mathcal{N}_d$ , we have*

$$\|f - P_{\underline{\kappa}}^{(\epsilon \underline{n})} f\|_\infty \lesssim \left( \prod_{i=1}^d \ln(n_i + 1) \right) \sum_{j=1}^d \frac{1}{(n_j + 1)^{s_j}} \omega \left( \frac{\partial^{s_j} f}{\partial x_j^{s_j}}; 0, \dots, 0, \frac{1}{n_j + 1}, 0, \dots, 0 \right),$$

*where*

$$\omega(f; \underline{u}) = \sup_{\substack{\underline{x}, \underline{x}' \in [-1, 1]^d \\ \forall i \in \{1, \dots, d\}: |x'_i - x_i| \leq u_i}} |f(\underline{x}') - f(\underline{x})|$$

*denotes the modulus of continuity of  $f$  on  $[-1, 1]^d$  (see [23, Section 6.3]).*

*Proof.* In view of (3.16), we only need to give a proper estimate of the best approximation  $E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f)$ . Since  $\underline{\Gamma}_{\underline{\kappa}, 0}^{(\epsilon \underline{n})} \subseteq \underline{\Gamma}_{\underline{\kappa}}^{(\epsilon \underline{n})}$ , we have  $E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f) \leq E_{\underline{\kappa}, 0}^{(\epsilon \underline{n})}(f)$ , where  $E_{\underline{\kappa}, 0}^{(\epsilon \underline{n})}(f)$  denotes the error of the best approximation in the space spanned by  $T_{\underline{\gamma}}$ ,  $\underline{\gamma} \in \underline{\Gamma}_{\underline{\kappa}, 0}^{(\epsilon \underline{n})}$ . Since  $\underline{\Gamma}_{\underline{\kappa}, 0}^{(\epsilon \underline{n})}$  has a tensor-product structure, we obtain

$$\begin{aligned} E_{\underline{\kappa}}^{(\epsilon \underline{n})}(f) &\leq E_{\underline{\kappa}, 0}^{(\epsilon \underline{n})}(f) \lesssim \sum_{j=1}^d \frac{2^{s_j} \omega \left( \frac{\partial^{s_j} f}{\partial x_j^{s_j}}; 0, \dots, 0, \frac{2}{\epsilon n_j + 1}, 0, \dots, 0 \right)}{(\epsilon n_j + 1)^{s_j}} \\ &\lesssim \sum_{j=1}^d \frac{\omega \left( \frac{\partial^{s_j} f}{\partial x_j^{s_j}}; 0, \dots, 0, \frac{1}{n_j + 1}, 0, \dots, 0 \right)}{(n_j + 1)^{s_j}} \end{aligned}$$

by using the estimates from [23, Section 5.3.2].  $\square$

Similar as stated in [19, Theorem 4.1] for the tensor-product case, we can also give a Dini-Lipschitz criterion for the uniform convergence of the error  $\|f - P_{\underline{\kappa}}^{(\epsilon \underline{n})} f\|_\infty$ .

If  $f \in C([-1, 1]^d)$  satisfies the condition

$$\omega(f; \underline{u}) \prod_{i=1}^d \ln u_i \rightarrow 0 \quad \text{as} \quad \underline{u} \rightarrow \underline{0},$$

then the polynomials  $P_{\underline{\kappa}}^{(\underline{u})} f$  converge in the  $L^\infty$ -norm to  $f$  as  $\min_{i=1, \dots, d} n_i \rightarrow \infty$ .

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